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TOWARDS AN INTERSECTION HOMOLOGY THEORY FOR REAL ALGEBRAIC VARIETIES

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ABSTRACT. This note considers equivariant intersection homology of stratified spaces with an involution. Specialisation gives a good intersection homology theory with 2-torsion coefficients of the set of fixed points, but no grading. To get a degree filtration, we consider the equivariant cohomology sheaves on the quotient space with respect to the corresponding perverse t-structure.

For algebraic varieties over the real numbers that admit a small resolution, it is shown that this procedure indeed provides the desired middle intersection homology theory, which even comes with a natural grading. In particular, it follows that the 2-torsion homology of a small resolution of a real algebraic variety is independent of the small resolution.

1. INTRODUCTION

In Borel's 1984 seminar on intersection cohomology, Goresky and MacPherson posed the problem whether there is a self-dual $\mathbf{Z}/2$-generalisation of intersection homology for real algebraic varieties. Apart from self-duality, the main criterion should be that if a variety has a small resolution, then the intersection homology should agree with the homology of the resolution.

They give an example of singular curves to show that this homology theory would not be a purely topological invariant. An example of a Schubert variety shows that even when the real algebraic variety is normal (hence a pseudo-manifold), such an intersection homology cannot coincide with standard intersection homology.

Since a real algebraic variety is the fixed point set of complex conjugation acting on the complex points of an algebraic variety defined over the real numbers, the natural thing to do is to try to define the intersection homology of the real points in terms of the topology of the complex points with the involution. This will give a topological invariant of the set of real points together with the action of complex conjugation on a small neighbourhood of the real points inside the complex points.

In Section 3 of this note we will see that indeed the localisation techniques of equivariant cohomology transform the $\mathbf{Z}/2$-valued intersection homology of the complex points into a $\mathbf{Z}/2$-valued homology theory of the real part with the above properties, except that this homology theory does not come with a natural grading.

In Section 4 we attempt to get a good grading by taking the filtration associated to a spectral sequence that computes equivariant cohomology in terms of the cohomology of perverse equivariant cohomology sheaves on the quotient space. Since the quotient space does not admit a good stratification with even-dimensional strata, we get *a priori* two filtrations, one for the lower and one for the upper middle perversity.

When our variety admits a small resolution, we get much better results. It should be said that small resolutions only exist in special situations, and when they exist they need not be unique; on the other hand, they do occur quite frequently in practice (e.g., in threefold theory and in the theory of Schubert varieties). Having a small resolution, it can be shown that both filtrations on our ungraded intersection homology agree with the degree filtration on the homology of the resolution. The construction then even gives an intrinsic grading, which coincides with the grading on the homology of the resolution. Hence in this case we get the self-dual graded homology theory we are after. In particular, this gives a proof of the fact that the $\mathbf{Z}/2$-valued homology of a small resolution of a real algebraic variety is independent of the small resolution (Corollary 4.10).

Whether the degree filtrations associated to the upper middle and lower middle perversity coincide for arbitrary real algebraic varieties remains an open question.

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2. LOCALISATION AND SPECIALISATION OF EQUIVARIANT COHOMOLOGY

This section contains a brief review of the theory of localisation of equivariant cohomology of spaces with an involution.

Let X be a reasonable finite-dimensional (but not necessarily compact) topological space with an involution $\sigma\colon X \to X$. We will denote the transformation group $\{1,\sigma\}$ by G. The inclusion of the fixed point set is denoted by

$$\iota\colon X^G \hookrightarrow X,$$

and the quotient map is denoted by

$$\pi\colon X \to X/G.$$

We are interested in G-equivariant cohomology. For this we will work in the derived category $D^b_G(X,\mathbf{Z}/2)$ of bounded complexes of G-sheaves of $\mathbf{Z}/2$-modules. Since G is finite, it is easiest to take G-sheaves in the 'naive' sense, as in [Gr, Ch. V] (but the more general construction of Bernstein and Lunts of $D^b_G(X,\mathbf{Z}/2)$ for an arbitrary compact Lie group G action gives the same result).

In any case, for an equivariant continuous map $f\colon X \to Y$ we have the usual pairs of adjoint functors (f^*, f_*) and $(f_!, f^!)$. In $D^b_G(X,\mathbf{Z}/2)$ we have the internal tensor product $\otimes$, the internal homomorphisms $\mathscr{H}om(-,-)$, the Verdier dualising sheaf $\mathfrak{D}_X := \phi^!\mathbf{Z}/2$, where $\phi\colon X \to \mathbf{pt}$ is the constant map, and the dualising functor

$$\mathfrak{D}_X(-) := \mathscr{H}om(-,\mathfrak{D}_X).$$

All of these functors have the usual properties. We also have some specific equivariant functors:

$$\begin{aligned}
\Gamma^G(X,-)\colon\ & D^b_G(X,\mathbf{Z}/2)\to D^b(\mathbf{Z}/2) && \text{(derived global equivariant sections)}\\
\pi^G_*\colon\ & D^b_G(X,\mathbf{Z}/2)\to D^b(X/G,\mathbf{Z}/2) && \text{(derived equivariant local sections)}\\
\pi^*_G\colon\ & D^b(X/G,\mathbf{Z}/2)\to D^b_G(X,\mathbf{Z}/2) && \text{(left adjoint of } \pi^G_*)
\end{aligned}$$

with the obvious relations $\Gamma^G(X,-)=\Gamma^G\circ\Gamma(X,-)=\Gamma(X/G,-)\circ\pi^G_*$. For a complex $\mathscr{F}$ of G-equivariant sheaves on X we write

$$H^i(X;G,\mathscr{F}):=H^i(\Gamma^G(X,\mathscr{F})).$$

The relation $\Gamma^G(X,-)=\Gamma^G\circ\Gamma(X,-)$ gives us a spectral sequence

$$E_2^{p,q}=H^p(G,H^q(X,\mathscr{F}))\Rightarrow H^{p+q}(X;G,\mathscr{F}),$$

which has many names in different context; here we will call it the Borel–Hochschild–Serre spectral sequence. We also have the analogues for sections with compact supports, denoted by Γ_c as usual.

On the complement of the fixed point set the group G acts freely, so there the functor π^G_* induces an equivalence of categories $D^b_G(X-X^G,\mathbf{Z}/2)\simeq D^b((X-X^G)/G,\mathbf{Z}/2)$ with inverse π^*_G. Finiteness of the cohomological dimension of X and $X-X^G$ implies the following localisation theorem.

Theorem 2.1. *For every $\mathscr{F}\in D^b_G(X,\mathbf{Z}/2)$ we have an $N>0$ such that the inclusion $\iota\colon X^G\to X$ induces isomorphisms*

(i) $H^k(X;G,\mathscr{F})\xrightarrow{\sim}H^k(X^G;G,\iota^*\mathscr{F})$ *for all $k>N$,*
(ii) $H^k(X^G;G,\iota^!\mathscr{F})\xrightarrow{\sim}H^k(X;G,\mathscr{F})$ *for all $k>N$.*

Since the cohomology ring $H^*(G,\mathbf{Z}/2)$ of G is isomorphic to the polynomial ring $\mathbf{Z}/2[\eta]$, with $\eta\in H^1(G,\mathbf{Z}/2)$, we get the following Borel–Atiyah–Segal localisation theorem.

Corollary 2.2. *The inclusion $\iota\colon X^G\to X$ induces isomorphisms*

(i) $H^*(X;G,\mathscr{F})\otimes_{\mathbf{Z}/2[\eta]}\mathbf{Z}/2[\eta,\eta^{-1}]\xrightarrow{\sim}H^*(X^G;G,\iota^*\mathscr{F})\otimes_{\mathbf{Z}/2[\eta]}\mathbf{Z}/2[\eta,\eta^{-1}]$
(ii) $H^*(X^G;G,\iota^!\mathscr{F})\otimes_{\mathbf{Z}/2[\eta]}\mathbf{Z}/2[\eta,\eta^{-1}]\xrightarrow{\sim}H^*(X;G,\mathscr{F})\otimes_{\mathbf{Z}/2[\eta]}\mathbf{Z}/2[\eta,\eta^{-1}]$.

If $\mathscr{E}$ is a complex of sheaves on X^G with trivial G-action, we have an isomorphism of graded $\mathbf{Z}/2[\eta]$-modules

$$H^*(X^G;G,\mathscr{E})\simeq H^*(X^G;\mathscr{E})\underset{\mathbf{Z}/2}{\otimes}\mathbf{Z}/2[\eta],$$

so writing

$$H^*(X^G;G,\mathscr{E})/(\eta-1):=H^*(X^G;G,\mathscr{E})\underset{\mathbf{Z}/2[\eta]}{\otimes}\mathbf{Z}/2[\eta]/(\eta-1),$$

we get that

$$H^*(X^G;G,\mathscr{E})/(\eta-1)=H^*(X^G;\mathscr{E}),$$

and the the Localisation Theorem implies the following 'specialisation' result:

Corollary 2.3. *Let $\mathscr{F} \in D^b_G(X,\mathbf{Z}/2)$ be a complex of G sheaves.*

(i) *If $\iota^*\mathscr{F}$ is quasi-isomorphic to a complex of sheaves on X^G with a trivial G-action, we have an isomorphism of $\mathbf{Z}/2$-modules*

$$H^*(X;G,\mathscr{F})/(1-\eta) \simeq H^*(X^G,\iota^*\mathscr{F}).$$

(ii) *If $\iota^!\mathscr{F}$ is quasi-isomorphic to a complex of sheaves on X^G with a trivial G-action, we have an isomorphism of $\mathbf{Z}/2$-modules*

$$H^*(X;G,\mathscr{F})/(1-\eta) \simeq H^*(X^G,\iota^!\mathscr{F}).$$

Corollary 2.4. *We have natural isomorphisms of $\mathbf{Z}/2$-modules*

$$H^*(X;G,\mathbf{Z}/2)/(1-\eta) \simeq H^*(X^G,\mathbf{Z}/2),$$
$$H^*(X;G,\mathfrak{D}_X(\mathbf{Z}/2))/(1-\eta) \simeq H^*(X^G,\mathfrak{D}_{X^G}(\mathbf{Z}/2)) \simeq H_*(X^G,\mathbf{Z}/2)$$

Note that here we denote homology with closed supports (often called 'Borel–Moore' homology) by $H_*(-,\mathbf{Z}/2)$; homology with compact supports (isomorphic to the usual singular homology) will be denoted by $H^c_*(-,\mathbf{Z}/2)$.

An important remark is that the grading on $H^*(X;G,\mathscr{F})$ does not induce a grading on $H^*(X;G,\mathscr{F})/(1-\eta)$, since the ideal $(1-\eta)$ is not homogeneous. Indeed, the group $H^*(X;G,\mathscr{F})/(1-\eta)$ is canonically isomorphic to the group $H^k(X;G,\mathscr{F})$ for any large enough k. In particular, this means that in the above circumstances we do not automatically recover the grading on $H^*(X^G,\iota^*\mathscr{F})$ from the grading on $H^*(X;G,\mathscr{F})$.

For example, if X is a smooth manifold of pure dimension n, then Corollary 2.3 gives two isomorphisms between $H^*(X;G,\mathbf{Z}/2)/(1-\eta)$ and $H^*(X^G,\mathbf{Z}/2)$; one via the equality $i^*\mathbf{Z}/2_X = \mathbf{Z}/2_{X^G}$, and one via the equality $\iota^!\mathbf{Z}/2_X = \oplus_{V\subset X^G}\mathbf{Z}/2_V[-\operatorname{codim}(V\subset X)]$. In general, the corresponding automorphism of $H^*(X^G,\mathbf{Z}/2)$ is not the identity, nor does it preserve the grading: it can be shown to be the cup product with the total Stiefel–Whitney class of the normal bundle of X^G in X (compare [DIKh, § 2.4]).

Remark. Theorem 2.1 and its corollaries also hold with compact supports. In the next sections we will use notation like $H^*_{(c)}$ to indicate results that are valid with closed as well as compact supports.

3. Intersection homology of fixed point sets

In the previous section we saw that we could recover the homology and cohomology of the fixed point set from the equivariant homology and cohomology of the total space, although we lost the information about the grading. This suggests that when X is a pseudo-manifold with an involution, we can define an intersection homology (with perversity $\mathbf{p}$) for X^G by specialising the equivariant cohomology of the intersection sheaf complex $\mathbf{IC}_\mathbf{p}(X,\mathbf{Z}/2)$ at $(1-\eta)$.

In order to stress the fact that this construction does *not* give any grading, we will use the notation

$$IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2) := H^*(X;G,\mathbf{IC}_\mathbf{p}(X,\mathbf{Z}/2))/(1-\eta).$$

Similarly we define fixed point set intersection homology with compact supports $IH^{\mathbf{p},c}_{\circledast}(X^G,\mathbf{Z}/2)$ by taking cohomology with compact supports on the right hand side. In both cases the result will not just depend on the topological space X^G, but on a small neighbourhood of X^G inside X together with the involution.

Remark 3.1. There are several conventions regarding the degrees in which the intersection complex **IC** is placed. In this section I follow [GM] by adopting the homological convention that **IC** is isomorphic to $\mathfrak{D}_X$ when X is a topological manifold. In other words, for arbitrary X the intersection homology is defined by $IH^{\mathbf{p}}_k(X,\mathbf{Z}/2) = H^{-k}(X,\mathbf{IC_p}(X,\mathbf{Z}/2))$.

By construction, the standard results on intersection homology for X get transported to our specialised equivariant intersection homology of X^G; in particular, we get the desired properties for real algebraic varieties as mentioned in the introduction.

3.1. **Arbitrary perversities.**

Theorem 3.2. *Let X be an n-dimensional pseudomanifold with an involution. For any perversities $\mathbf{p} \le \mathbf{q}$ we have natural maps*

$$H^{\circledast}_{(c)}(X^G,\mathbf{Z}/2) \to IH^{\mathbf{p},(c)}_{\circledast}(X^G,\mathbf{Z}/2) \to IH^{\mathbf{q},(c)}_{\circledast}(X^G,\mathbf{Z}/2) \to H^{(c)}_{\circledast}(X,\mathbf{Z}/2),$$

with the following properties:

(i) *The composite map is cap product with the total fundamental class*

$$\mu^{\circledast}_{X^G} \in H_{\circledast}(X^G,\mathbf{Z}/2)$$

which is by definition the congruence class modulo $(1-\eta)$ of the equivariant fundamental class $\mu^G_X \in H^{-n}(X;G,\mathfrak{D}_X(\mathbf{Z}/2))$.

(ii) *All maps are isomorphisms when X is a $\mathbf{Z}/2$-homology manifold.*

Proof. The natural maps in $D^b(X,\mathbf{Z}/2)$

$$\mathbf{Z}/2[n] \to \mathbf{IC_p} \to \mathbf{IC_q} \to \mathfrak{D}_X(\mathbf{Z}/2)$$

(see [GM, Prop. 5.1, §5.5]) are G-equivariant, and they are all quasi-isomorphisms when X is a $\mathbf{Z}/2$-homology manifold. The constructions and the last part of the theorem follow immediately. ☐

For $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ such that $\mathbf{p}+\mathbf{q} \le \mathbf{r}$, we have natural pairings

$$\begin{array}{ccccc} IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2) & \otimes & IH^{\mathbf{q}}_{\circledast}(X^G,\mathbf{Z}/2) & \to & IH^{\mathbf{r}}_{\circledast}(X^G,\mathbf{Z}/2) \\ IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2) & \otimes & IH^{\mathbf{q},c}_{\circledast}(X^G,\mathbf{Z}/2) & \to & IH^{\mathbf{r},c}_{\circledast}(X^G,\mathbf{Z}/2) \end{array}$$

Theorem 3.3. *Let X be an n-dimensional pseudomanifold with an involution and let $\mathbf{p}$ and $\mathbf{q}$ be complementary perversities. The above pairing and the trace map $\Gamma_c(X,\mathbf{IC_t}) \to \mathbf{Z}/2$ induce a perfect pairing*

$$IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2) \otimes IH^{\mathbf{q},c}_{\circledast}(X^G,\mathbf{Z}/2) \to \mathbf{Z}/2.$$

Proof. Since the composite map

$$\mathbf{IC_p} \otimes \mathbf{IC_q} \to \mathbf{IC_t}[n] \to \mathfrak{D}_X[n]$$

is a Verdier dual pairing by [GM, Th. 5.3], we get that the induced map

$$\Gamma(X, \mathbf{IC_p}) \to \mathscr{H}om(\Gamma_c(X, \mathbf{IC_q}), \mathbf{Z}/2[n])$$

is an isomorphism, so the result follows from Lemma 3.4 below. □

Lemma 3.4. *With notation as above, let $\mathscr{M}$ be a bounded complex of G-equivariant $\mathbf{Z}/2$-modules. Then the canonical pairing*

$$H^*(G, \mathscr{M}) \otimes H^*(G, \mathscr{H}om(\mathscr{M}, \mathbf{Z}/2)) \to H^*(G, \mathbf{Z}/2)$$

induces a perfect pairing

$$H^*(G, \mathscr{M})/(1-\eta) \otimes H^*(G, \mathscr{H}om(\mathscr{M}, \mathbf{Z}/2))/(1-\eta) \to \mathbf{Z}/2.$$

Proof. This is the 'hypercohomology' version of the standard duality in the cohomology of G. □

3.2. **Middle perversity.** If X is a $\mathbf{Z}/2$-Witt space (as in [GM, §5.6]) it admits a middle intersection sheaf $\mathbf{IC}(X, \mathbf{Z}/2)$, hence we get ungraded specialised middle intersection homology groups for X^G with closed and compact supports by putting

$$IH^{(c)}_{\circledast}(X^G, \mathbf{Z}/2) := H^*_{(c)}(X; G, \mathbf{IC}(X, \mathbf{Z}/2))/(1-\eta).$$

Again, by construction it inherits all the usual properties from the intersection homology of X:

Theorem 3.5. *With notation as above, the intersection pairing*

$$IH_{\circledast}(X^G, \mathbf{Z}/2) \otimes IH^{c}_{\circledast}(X^G, \mathbf{Z}/2) \to \mathbf{Z}/2.$$

is perfect.

Proof. Immediate from Theorem 3.3. □

Small maps and resolutions. Recall that in [GM] a proper surjective morphism $f\colon Y \to X$ of (not necessarily complete) irreducible N-dimensional complex algebraic varieties is called *homologically small* if for all $q > 0$ the locus of points

$$\{x \in X(\mathbf{C})\colon\ \mathscr{H}^{q-2N}(f_* \mathbf{IC}(Y(\mathbf{C}), \mathbf{Z}/2))_x \neq 0\}$$

has algebraic codimension $> q$. In particular f is finite over a Zariski-open $U \subset X$ and we define the degree of f as the degree of f over U. A normalisation map is homologically small, and so is a *small resolution*: a proper surjective morphism $f\colon X \to Y$ of irreducible varieties, such that Y is smooth, f is a birational isomorphism, and for every $r > 0$ the locus $\{x \in X\colon\ \dim f^{-1}(x) \geq r\}$ has codimension $> 2r$.

We will be interested in the case where X, Y and f are defined over $\mathbf{R}$, so that f is equivariant with respect to G acting via complex conjugation on $X(\mathbf{C})$ and $Y(\mathbf{C})$. More generally, we can consider any continuous G-action on $X(\mathbf{C})$ and $Y(\mathbf{C})$ (with respect to the Euclidean topology). By slight abuse of terminology and notation we will say that G acts via a *continuous* involution on X and Y.

Theorem 3.6. *Let $f\colon Y \to X$ be a G-equivariant homologically small map of degree 1 between complex algebraic varieties with a continuous involution. Then*

$$IH^{(c)}_{\circledast}(Y(\mathbf{C})^G, \mathbf{Z}/2) \simeq IH^{(c)}_{\circledast}(X(\mathbf{C})^G, \mathbf{Z}/2)$$

Proof. We have that $f_*\mathbf{IC}(Y(\mathbf{C}),\mathbf{Z}/2) \simeq \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)$ by [GM, Th. 6.2]. □

Corollary 3.7. *If $f\colon Y \to X$ is a G-equivariant small resolution of a complex algebraic variety with a continuous involution, then*

$$IH^{(c)}_{\circledast}(X(\mathbf{C})^G, \mathbf{Z}/2) \simeq H^{(c)}_{*}(Y(\mathbf{C})^G, \mathbf{Z}/2).$$

Künneth formula. In view of [GM, Prop. 6.3], a Künneth formula for fixed point set middle intersection homology follows from the fact that

$$H^*(G, \mathscr{M} \otimes \mathscr{N})/(1-\eta) \simeq H^*(G, \mathscr{M})/(1-\eta) \otimes H^*(G, \mathscr{N})/(1-\eta)$$

for bounded complexes $\mathscr{M}$, $\mathscr{N}$ of G-equivariant $\mathbf{Z}/2$-vector spaces.

4. A degree filtration

In this section we will show that for any perversity $\mathbf{p}$ our specialised equivariant intersection homology group $IH^{\mathbf{p}}_{\circledast}(X^G, \mathbf{Z}/2)$ admits a filtration that can be considered as a degree filtration.

This filtration comes from a $\mathbf{p}$-perverse version of the Grothendieck spectral sequence associated to the composition of derived functors $\Gamma^G_X = \Gamma_{X/G} \circ \pi^G_*$. For this we use the $\mathbf{p}$-perverse t-structure on the derived category of sheaves on X/G.

Then we analyse the case of an algebraic variety X over the real numbers that admits a small resolution. There we see that the middle intersection complex $\pi^G_* \mathbf{IC}(X(\mathbf{C}), \mathbf{Z}/2)$ actually splits (up to a bounded mapping cone) into a direct sum of shifted copies of a complex of sheaves $\mathbf{IC}(X(\mathbf{R}), \mathbf{Z}/2)$ on $X(\mathbf{R})$ which is (up to a shift) perverse for both the upper and the lower middle perversity. The splitting provides a grading on $IH_{\circledast}(X(\mathbf{R}), \mathbf{Z}/2)$ compatible (up to a shift) with our degree filtrations for both the upper and the lower middle perversity.

4.1. Strictly G-equivariant stratifications and perverse t-structures.

Definition 4.1. Let X be a topological space with an action of $G = \{1, \sigma\}$. A *stratification* $\mathscr{S}$ of X will be a finite partition of X in locally closed subspaces with the following properties (cf. [BBD, 2.1.13]).

- Each stratum $S \in \mathscr{S}$ is a topological manifold where every connected component has the same (finite) dimension.
- The boundary of each stratum is a union of strata of smaller dimensions.
- For any $i_S\colon S \hookrightarrow X$ the functor $(i_S)_*$ has finite cohomological dimension and for any locally constant sheaf $\mathscr{F}$ of $\mathbf{Z}/2$-modules of finite rank, the $\mathscr{H}^n((i_S)_*\mathscr{F})$ are locally constant along any $S' \in \mathscr{S}$.

We say that a stratification $\mathscr{S}$ is *strictly G-equivariant* if it also satisfies the following properties.

- For each $S \in \mathscr{S}$ we have $\sigma(S) = S$ and either $S^G = S$ or $S^G = \emptyset$.

A *stratified G-space* will be a topological space with an action of G and a strictly G-equivariant stratification. Clearly, for any strictly G-equivariant stratification of X we have that $\mathscr{S}/G := \{S/G\colon\ S \in S\}$ gives a stratification of X/G, and $\mathscr{S}^G := \{S \in \mathscr{S}\colon\ S^G = S\}$ gives a stratification of X^G which both satisfy the properties of [BBD, 2.1.13]. For any $\mathscr{S}$-constructible complex of G-sheaves $\mathscr{C}$ on X we have that $\pi_*\mathscr{C}$ is $\mathscr{S}/G$ constructible and $i^*\mathscr{C}$ is $\mathscr{S}^G$ constructible. Of course, neither X/G nor X^G will in general be a pseudomanifold in the sense of [GM], but this does not matter here.

As usual $\mathbf{p}\colon\ \mathscr{S} \to \mathbf{Z}$ will be a perversity such that $\mathbf{p}(S)$ only depends on the dimension of S. Since we use the BBD-formalism in this section, our special perversities are:

- The zero perversity $\mathbf{0}\colon\ S \mapsto 0$.
- The lower middle perversity $\lfloor \mathbf{m} \rfloor\colon\ S \mapsto -\lfloor \dim S/2 \rfloor$
- The upper middle perversity $\lceil \mathbf{m} \rceil\colon\ S \mapsto -\lceil \dim S/2 \rceil = -\lfloor (\dim S+1)/2 \rfloor$.
- The top perversity $\mathbf{t}\colon\ S \mapsto -\dim S$.

Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$ and $\lceil x \rceil$ denotes the smallest integer $\geq x$. Observe that after the usual reindexing, the lower middle perversity $\lfloor \mathbf{m} \rfloor$ actually corresponds to the lower middle perversity $\bar{m}$ of [GM] if X is odd dimensional and to the upper middle perversity $\bar{n}$ of *loc. cit.* if X is even dimensional. The *dual* $\mathbf{p}^*$ of a perversity $\mathbf{p}$ is defined by $\mathbf{p}^* = \mathbf{t} - \mathbf{p}$.

Recall that the $\mathbf{p}$-perverse t-structure on $D(X, \mathbf{Z}/2)$ is given by

$$
\begin{aligned}
{}^{\mathbf{p}}D^{\leq 0}(X, \mathbf{Z}/2) &= \{\mathscr{C}\colon\ \mathscr{H}^n i_S^* \mathscr{C} = 0 \text{ for } S \in \mathscr{S},\, n > \mathbf{p}(S)\} \\
{}^{\mathbf{p}}D^{\geq 0}(X, \mathbf{Z}/2) &= \{\mathscr{C}\colon\ \mathscr{H}^n i_S^! \mathscr{C} = 0 \text{ for } S \in \mathscr{S},\, n < \mathbf{p}(S)\}
\end{aligned}
$$

and similarly on X/G and X^G. The heart of this t-structure is the category of $\mathbf{p}$-perverse sheaves (of $\mathbf{Z}/2$-modules) on X. The cohomology sheaves associated to this t-structure will be denoted by ${}^{\mathfrak{p}}\mathscr{H}^*(\mathscr{C})$. For the perversities mentioned above, the t-structure on the subcategory of $\mathscr{S}$-constructible sheaves does not change when we refine $\mathscr{S}$ (see [BBD, Prop. 2.1.14]).

Observe that $\iota_*\colon\ D(X^G, \mathbf{Z}/2) \to D(X, \mathbf{Z}/2)$, $\iota_*\colon\ D(X^G, \mathbf{Z}/2) \to D(X/G, \mathbf{Z}/2)$, and $\pi_*\colon\ D(X, \mathbf{Z}/2) \to D(X/G, \mathbf{Z}/2)$ are p-exact.

4.2. **Specialisation and perverse t-structures.** A key property of the cohomology of $G = \mathbf{Z}/2$ that is used in equivariant localisation is the fact that the nontrivial cohomology class $\eta \in H^1(G, \mathbf{Z}/2) = \mathbf{Z}/2$ induces for every G-module M and every $n > 0$ an isomorphism $H^n(G, M) \simeq H^{n+1}(G, M)$.

In the context of derived categories, η gives for every bounded complex of G-sheaves $\mathscr{C}$ on X a morphism of unbounded complexes $\pi_*^G(\mathscr{C}) \to \pi_*^G(\mathscr{C})[1]$ of sheaves on X/G such that the mapping cone is a bounded complex.

Lemma 4.2. *For every bounded complex of G-sheaves $\mathscr{C}$ on X there is an $N \in \mathbf{Z}$ such that the canonical map $\pi_*^G(\mathscr{C}) \to \pi_*^G(\mathscr{C})[1]$ induces an isomorphism ${}^{\mathfrak{p}}\mathscr{H}^n(\pi_*^G\mathscr{C}) \simeq {}^{\mathfrak{p}}\mathscr{H}^{n+1}(\pi_*^G\mathscr{C})$ for every $n > N$ and any perversity $\mathbf{p}$.*

Proof. Applying the homological functor ${}^{\mathfrak{p}}\mathscr{H}^*$ to the distinguished triangle $\pi_*^G(\mathscr{C}) \to \pi_*^G(\mathscr{C})[1] \to \mathit{Cone}$, we see that this follows from the fact that the mapping cone is bounded and that for any bounded complex of sheaves ${}^{\mathbf{p}}\mathscr{H}^n$ vanishes for n large enough. □

Hence for any bounded complex of G-sheaves $\mathscr{C}$ on X we may define

$$ {}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C} := {}^{\mathbf{p}}\mathscr{H}^n(\pi^G_*\mathscr{C}) $$

for n large enough. The key result that will be used in this paper to prove things about perverse sheaves of the form ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}$ is the following observation.

Lemma 4.3. *For any homomorphism $\mathscr{C} \to \mathscr{C}'$ of bounded complexes of G-sheaves on X and any perversity $\mathbf{p}$ we have that*

$$ {}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C} \xrightarrow{\sim} {}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}' $$

if and only if

$$ \mathscr{H}^{\infty}_G\mathscr{C} \xrightarrow{\sim} \mathscr{H}^{\infty}_G\mathscr{C}' $$

Proof. Either condition is equivalent to the fact that the mapping cone of $\pi^G_*\mathscr{C} \to \pi^G_*\mathscr{C}'$ is a bounded complex. □

Corollary 4.4. *The perverse sheaf ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}$ has supports in X^G and*

$$ {}^{\mathbf{p}}\mathscr{H}^{\infty}_G(\iota^!\mathscr{C}) = {}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C} = {}^{\mathbf{p}}\mathscr{H}^{\infty}_G(\iota^*\mathscr{C}). $$

Proof. Immediate from Lemma 4.3 and the sheaf-theoretic version of Theorem 2.1. □

In other words, ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}$ can be considered as an equivariant specialisation of the sheaf $\mathscr{C}$, even though in general $H^*(X;G,\mathscr{C})/(1-\eta)$ will not be isomorphic to $H^*(X^G,{}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C})$.

Corollary 4.5. *Let $\mathscr{C}$ be a bounded complex of G-sheaves on X. If we have a $\mathbf{p}$-perverse sheaf $\mathscr{P}$ supported on X^G (with a trivial G-action) and a morphism $\mathscr{P} \to \mathscr{C}$ or $\mathscr{C} \to \mathscr{P}$ such that $\mathscr{H}^{\infty}_G\mathscr{C} \simeq \mathscr{H}^{\infty}_G\mathscr{P}$, then $\mathscr{P} \simeq {}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}$ and $H^*(X^G,\mathscr{P}) \simeq H^*(X^G,{}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathscr{C}) \simeq H^*(X;G,\mathscr{P})/(1-\eta)$.*

Proof. Since G acts trivially on $\mathscr{P}$, we have that $\pi^G_*\mathscr{P} = \oplus_{i\geq 0}\mathscr{P}[-i]$. Since $\mathscr{P}$ is $\mathbf{p}$-perverse, ${}^{\mathbf{p}}\mathscr{H}^n$ of the right hand term is $\mathscr{P}$ for every $n \geq 0$. Now apply Lemma 4.3. □

Corollary 4.6. *Let $\mathscr{C}$ be a complex of G-sheaves on X, then ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathfrak{D}_X(\mathscr{C}) \simeq \mathfrak{D}_{X^G}({}^{\mathbf{p}^*}\mathscr{H}^{\infty}_G\mathscr{C})$.*

Proof. By Corollary 4.4 and the general properties of Verdier duality, it is sufficient to prove that ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathfrak{D}_{X^G}(i^*\mathscr{C}) \simeq \mathfrak{D}_{X^G}({}^{\mathbf{p}^*}\mathscr{H}^{\infty}_G i^*\mathscr{C})$.

Let $\mathbf{Z}/2[G]$ be the group ring of G. Considering $i^*\mathscr{C}$ as a sheaf of $\mathbf{Z}/2[G]$-modules, we get that ${}^{\mathbf{p}}\mathscr{H}^{\infty}_G\mathfrak{D}_{X^G}(i^*\mathscr{C}) = {}^{\mathbf{p}}\mathscr{H}^{\infty}\mathscr{H}om_{\mathbf{Z}/2[G]}(i^*\mathscr{C},\mathfrak{D}_{X^G}) = {}^{\mathbf{p}}\mathscr{H}^{\infty}\mathfrak{D}_{X^G}(i^*\mathscr{C}\otimes_{\mathbf{Z}/2[G]}\mathbf{Z}/2)$ On the other hand, by duality of perversity we get $\mathfrak{D}_{X^G}({}^{\mathbf{p}^*}\mathscr{H}^{\infty}_G i^*\mathscr{C}) = {}^{\mathbf{p}}\mathscr{H}^{-\infty}\mathfrak{D}_{X^G}(\mathscr{H}om_{\mathbf{Z}/2[G]}(\mathbf{Z}/2,i^*\mathscr{C}))$.

Hence the statement reduces to the claim that for very large N we have ${}^{\mathbf{p}}\mathscr{H}^N\mathfrak{D}_{X^G}(i^*\mathscr{C}\otimes_{\mathbf{Z}/2[G]}\mathbf{Z}/2) = {}^{\mathbf{p}}\mathscr{H}^{-N}\mathfrak{D}_{X^G}(\mathscr{H}om_{\mathbf{Z}/2[G]}(\mathbf{Z}/2,i^*\mathscr{C}))$

Since any bounded below (resp. above) complex of sheaves has bounded below (resp. above) $\mathbf{p}$-perverse t-structure, it is sufficient to prove that for any $r > 0$ and any large enough N we have $\tau_{[-N-r,-N+r]} i^*\mathscr{C} \otimes_{\mathbf{Z}/2[G]} \mathbf{Z}/2 = \tau_{[N-r,N+r]} \mathscr{H}om_{\mathbf{Z}/2[G]}(\mathbf{Z}/2, i^*\mathscr{C})[2N]$, and this follows from the standard duality and periodicity in the cohomology of $G = \mathbf{Z}/2$. □

4.3. **Construction of the filtration on specialised equivariant intersection homology.** Let X be a stratified G-space, let $\mathscr{C}$ be a bounded complex of G-sheaves of $\mathbf{Z}/2$-modules on X and let $\mathbf{p}$ be a perversity. Consider the Grothendieck spectral sequence

$$^{\mathbf{p}}E_2^{r,s}(\mathscr{C}) = H^r(X/G, {}^{\mathbf{p}}\mathscr{H}_G^s\mathscr{C}) \Rightarrow H^{r+s}(X;G,\mathscr{C}).$$

associated to the isomorphism of derived functors

$$\Gamma_X^G = \Gamma_{X/G} \circ \pi_*^G.$$

The spectral sequence gives for every n a finite filtration $\dots \subset F^r \subset F^{r-1} \subset \dots$ of the cohomology group $H^n(X;G,\mathscr{C})$, with rth graded piece equal to $^{\mathbf{p}}E_\infty^{r,n-r}(\mathscr{C})$. This filtration passes to the quotient $H^*(X;G,\mathscr{C})/(1-\sigma)$, since on the level of the spectral sequence the map ${}^{\mathbf{p}}\mathscr{H}_G\mathscr{C} \to {}^{\mathbf{p}}\mathscr{H}_G\mathscr{C}[1]$ associated to $\eta \in H^1(G,\mathbf{Z}/2)$ gives a map $^{\mathbf{p}}E^{r,s}(\mathscr{C}) \to {}^{\mathbf{p}}E^{r,s+1}(\mathscr{C})$. Observe that the rth graded piece of this filtration on $H^*(X;G,\mathscr{C})/(1-\sigma)$ is a subquotient of $H^r(X^G, {}^{\mathbf{p}}\mathscr{H}_G^\infty\mathscr{C})$.

If we apply this construction to the constant sheaf $\mathbf{Z}/2$ and the zero perversity, we get the degree filtration

$$F^r = \bigoplus_{i \geq r} H^i(X^G, \mathbf{Z}/2)$$

on $H^*(X;G,\mathbf{Z}/2)/(1-\sigma) = H^*(X^G,\mathbf{Z}/2)$. With $\mathscr{C} = \mathfrak{D}_X$ and $\mathbf{p} = \mathbf{t}$ we get the degree filtration

$$F^r = \bigoplus_{i \geq r} H^i(X^G, \mathfrak{D}_{X^G}) = \bigoplus_{i \leq -r} H_i(X^G, \mathbf{Z}/2).$$

By analogy we now apply this construction to $\mathscr{C} = \mathbf{IC_p}(X,\mathbf{Z}/2)$ to *define* a degree filtration on $IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2)$. We will see below that this gives the right result in at least one nontrivial situation: the case of the middle intersection homology of a real algebraic variety admitting a small resolution.

Remark 4.7. Up to a bounded mapping cone, the complex $\pi_*^G(\mathbf{Z}/2)_X$ actually decomposes into a direct sum of shifted cohomology sheaves $\bigoplus_{i \geq 0} \mathscr{H}_G^\infty \mathbf{Z}/2[-i] = \bigoplus_{i \geq 0} \mathbf{Z}/2_{X^G}[-i]$, and $\pi_*^G \mathfrak{D}_X$ decomposes into $\bigoplus_{i \geq 0} {}^{\mathbf{t}}\mathscr{H}_G^\infty \mathfrak{D}_X[-i] = \bigoplus_{i \geq 0} \mathfrak{D}_{X^G}[-i]$ up to a bounded mapping cone. A similar decomposition of the $\pi_*^G \mathbf{IC_p}$ into shifted copies of ${}^{\mathbf{p}}\mathscr{H}_G^\infty \mathbf{IC_p}$ would provide a grading on $IH^{\mathbf{p}}_{\circledast}(X^G,\mathbf{Z}/2)$ rather than just a degree filtration. We will see below that we have such a decomposition for the middle intersection complex on a variety over the real numbers that admits a small resolution, but I have no idea whether such a decomposition exists in general.

4.4. **Middle intersection homology.** When X is a $\mathbf{Z}/2$-Witt space, the upper middle and lower middle perversities give rise to a single middle intersection homology sheaf $\mathbf{IC}(X,\mathbf{Z}/2)$ and specialisation gives us a single ungraded middle intersection homology group $IH_{\circledast}(X^G,\mathbf{Z}/2)$. However, our filtration depends again on a choice for the upper or the lower middle perversity. Of course, one would hope that these two filtrations are essentially the same, but to me this seems unlikely without extra conditions on X and the involution.

In the setting of real algebraic geometry, the situation looks better. We will see below that the $\lfloor\mathbf{m}\rfloor$- and the $\lceil\mathbf{m}\rceil$-filtration coincide (possibly up to a shift) whenever there is a small resolution. For real algebraic varieties without small resolutions this remains an open problem.

4.5. **Small resolutions.** Let X be a (not necessarily complete) algebraic variety of pure dimension N defined over the real numbers. Recall from Section 3.2 that a small resolution $f\colon Y\to X$ is a proper surjective morphism (defined over the real numbers) such that Y is smooth, f is a birational isomorphism, and for every $r>0$ the locus $\{x\in X:\ \dim f^{-1}(x)\geq r\}$ has codimension $>2r$.

Lemma 4.8. *If $f\colon Y\to X$ is a small resolution of an algebraic variety of dimension N defined over the real numbers, then $f_*\mathbf{Z}/2_{Y(\mathbf{R})}[\lfloor N/2\rfloor]$ is an $\lfloor\mathbf{m}\rfloor$-perverse sheaf and $f_*\mathbf{Z}/2_{Y(\mathbf{R})}[\lceil N/2\rceil]$ is an $\lceil\mathbf{m}\rceil$-perverse sheaf on X*

Proof. Let $\mathscr{S}$ be a stratification of $X(\mathbf{R})$, such that $f_*\mathbf{Z}/2$ is $\mathscr{S}$-constructible. Let $S\in\mathscr{S}$ be a stratum of dimension $d<N$. By definition, for any $x\in S$, the fibre of $Y(\mathbf{R})\to X(\mathbf{R})$ over x has dimension $<(N-d)/2$, hence

$$\mathscr{H}^n(i_S^*f_*\mathbf{Z}/2)=0 \text{ for } n\notin[0,\lfloor(N-d-1)/2\rfloor]$$
$$\mathscr{H}^n(i_S^!f_*\mathbf{Z}/2)=0 \text{ for } n\notin[\lceil(N-d+1)/2\rceil,N-d]$$

For $i_S^*f_*\mathbf{Z}/2$ this follows from the proper base change theorem. For $i_S^!f_*\mathbf{Z}/2$ this follows from the fact that S is smooth of dimension d, $Y(\mathbf{R})$ is smooth of dimension N and f is proper, so that

$$i_S^!f_*\mathbf{Z}/2=i_S^!f_*\mathfrak{D}_{Y(\mathbf{R})}[-N]=\mathfrak{D}_S(i_S^*f_!\mathbf{Z}/2)[-N]=\mathscr{H}om(i_S^*f_*\mathbf{Z}/2,\mathbf{Z}/2)[d-N].$$

From the inequality $\lfloor a/2\rfloor-\lfloor b/2\rfloor\leq-\lfloor(b-a)/2\rfloor$ for $a,b\in\mathbf{Z}$ we deduce that $\lfloor(N-d-1)/2\rfloor-\lfloor N/2\rfloor\leq-\lfloor(d+1)/2\rfloor\leq-\lfloor d/2\rfloor$ and $\lfloor(N-d-1)/2\rfloor-\lfloor(N+1)/2\rfloor\leq-\lfloor(d+2)/2\rfloor\leq-\lfloor(d+1)/2\rfloor=-\lceil d/2\rceil$. It follows that

$$\mathscr{H}^n(i_S^*f_*\mathbf{Z}/2[\lfloor N/2\rfloor])=0 \text{ for } n>-\lfloor d/2\rfloor$$
$$\mathscr{H}^n(i_S^*f_*\mathbf{Z}/2[\lceil N/2\rceil])=0 \text{ for } n>-\lceil d/2\rceil.$$

Similarly, the equality $\lceil a/2\rceil-\lfloor b/2\rfloor=-\lfloor(b-a)/2\rfloor$ gives us that $\lceil(N-d+1)/2\rceil-\lfloor N/2\rfloor=-\lfloor(d-1)/2\rfloor\geq-\lfloor d/2\rfloor$ and $\lceil(N-d+1)/2\rceil-\lfloor(N+1)/2\rfloor=-\lfloor d/2\rfloor\geq-\lceil d/2\rceil$. It follows that

$$\mathscr{H}^n(i_S^!f_*\mathbf{Z}/2[\lfloor N/2\rfloor])=0 \text{ for } n<-\lfloor d/2\rfloor$$
$$\mathscr{H}^n(i_S^!f_*\mathbf{Z}/2[\lceil N/2\rceil])=0 \text{ for } n<-\lceil d/2\rceil.$$

□

Corollary 4.9. *Let $f\colon Y \to X$ be a small resolution of an algebraic variety over the real numbers of dimension N. Then*

(i) *The quasi-isomorphism $\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2) \simeq f_*\mathbf{Z}/2_{Y(\mathbf{C})}$ induces isomorphisms*

$$^{\lfloor \mathbf{m} \rfloor}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2) \simeq f_*\mathbf{Z}/2_{Y(\mathbf{R})}[\lfloor N/2 \rfloor]$$
$$^{\lceil \mathbf{m} \rceil}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2) \simeq f_*\mathbf{Z}/2_{Y(\mathbf{R})}[\lceil N/2 \rceil]$$

(ii) *We have natural isomorphisms*

$$IH^{(\mathrm{c})}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) \simeq H^*_{(\mathrm{c})}(X^G, {}^{\lfloor \mathbf{m} \rfloor}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$$
$$IH^{(\mathrm{c})}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) \simeq H^*_{(\mathrm{c})}(X^G, {}^{\lceil \mathbf{m} \rceil}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$$

compatible with the corresponding degree filtrations.

(iii) *The degree filtrations on $IH^{(\mathrm{c})}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2)$ corresponding to ${}^{\lfloor \mathbf{m} \rfloor}E^{r,s}_{(\mathrm{c})}(\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$ and ${}^{\lceil \mathbf{m} \rceil}E^{r,s}_{(\mathrm{c})}(\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$ coincide (up to a shift in degree by 1 if N is odd).*

Proof. Immediate from Lemma 4.8, the sheaf version of Corollary 3.7 and Corollary 4.5. □

This means that when X admits a small resolution $Y \to X$, we have an intrinsic definition of an intersection complex on the real part by writing

$$\mathbf{IC}(X(\mathbf{R}),\mathbf{Z}/2) := {}^{\lfloor \mathbf{m} \rfloor}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)[\lceil N/2 \rceil]$$
$$(= {}^{\lceil \mathbf{m} \rceil}\mathscr{H}_G^\infty \mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)[\lfloor N/2 \rfloor]).$$

Putting

$$IH^{(\mathrm{c})}_i(X(\mathbf{R}),\mathbf{Z}/2) := H^{-i}_{(\mathrm{c})}(X(\mathbf{R}),\mathbf{IC}(X(\mathbf{R}),\mathbf{Z}/2)),$$

we get an isomorphism $IH^{(\mathrm{c})}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) = IH^{(\mathrm{c})}_*(X(\mathbf{R}),\mathbf{Z}/2)$ and a graded isomorphism $IH^{(\mathrm{c})}_*(X(\mathbf{R}),\mathbf{Z}/2) \simeq H^{(\mathrm{c})}_*(Y(\mathbf{R}),\mathbf{Z}/2)$. Corollary 4.6 gives us the required nondegenerate pairing of graded $\mathbf{Z}/2$-modules

$$IH_*(X(\mathbf{R}),\mathbf{Z}/2) \times IH^{\mathrm{c}}_{N-*}(X(\mathbf{R}),\mathbf{Z}/2) \to \mathbf{Z}/2.$$

In particular, it follows that different small real algebraic resolutions have the same $\mathbf{Z}/2$-homology.

Corollary 4.10. *When $Y \to X$ and $Y' \to X$ are two small resolutions of an algebraic variety defined over the real numbers, then $H_*(Y(\mathbf{R}),\mathbf{Z}/2)$ and $H_*(Y'(\mathbf{R}),\mathbf{Z}/2)$ are isomorphic as graded $\mathbf{Z}/2$-vector spaces, and the same holds for homology with compact supports.*

Remark. After distributing the first version of this note, Parusinski kindly sent me a manuscript of a work in progress in which he proposes an explicit chain complex on any real algebraic variety which gives 2-torsion homology groups that are isomorphic to the homology groups of any small resolution. In particular, he obtains a different proof of Corollary 4.10. At this stage it is not clear whether Parusinki's ideas will lead to homology groups with a nondegenerate intersection product. A proof of Corollary 4.10 by completely different methods was announced by Totaro in [T].

CONCLUSION

For a (not necessarily complete) algebraic variety X defined over the real numbers we have introduced ungraded middle intersection homology $IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2)$ and $IH^{\mathrm{c}}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2)$ of the real part which statisfies the desired properties. These groups only depend on the topology of complex conjugation acting on a small neighbourhood of $X(\mathbf{R})$ inside $X(\mathbf{C})$.

The ungraded intersection homology comes with two natural degree filtrations, corresponding to the upper and lower middle perversity. In particular, this allows us to define upper and lower middle intersection Betti numbers of $X(\mathbf{R})$.

When X admits a small resolution, the two degree filtrations (hence the two sets of Betti numbers) coincide up to a shift. Moreover, in this case we actually get a compatible grading on $IH^{(\mathrm{c})}_{\circledast}$.

Questions that remain open for an algebraic variety X over the real numbers that does not admit a small resolution:

- Do we have that
$$IH^{(\mathrm{c})}_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) = H^{*}_{(\mathrm{c})}(X(\mathbf{R}), {}^{\mathbf{p}}\mathscr{H}^{\infty}_{G}\,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$$
for $\mathbf{p} = \lfloor \mathbf{m} \rfloor$ and $\mathbf{p} = \lceil \mathbf{m} \rceil$?
- Do ${}^{\lfloor \mathbf{m} \rfloor}\mathscr{H}^{\infty}_{G}\,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)$ and ${}^{\lceil \mathbf{m} \rceil}\mathscr{H}^{\infty}_{G}\,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)$ coincide (up to a shift)?
- Do the upper and lower middle intersection Betti numbers of $X(\mathbf{R})$ coincide (up to a shift)?

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