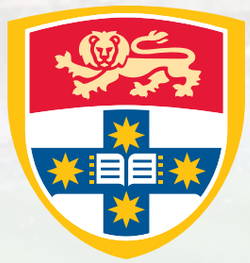
A nonlinear interpolation result with application to nonlinear semigroups

Joint work with Thierry Coulhon (Research University PSL, Paris)

PDE Seminar

Sydney University, Carslaw Building, 7 March 2016

Daniel Hauer



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Linear Interpolation Theory

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Linear Interpolation Theory

Let (Σ, μ) & (Ω, ν) be two σ-finite measure spaces.

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Linear Interpolation Theory

Let (Σ, μ) & (Ω, ν) be two σ-finite measure spaces.

For $1 \le p_0, p_1, q_0, q_1 \le \infty$ suppose

(2) $T : L^{p_0}(\mu) \longrightarrow L^{q_0}(\nu)$ linear bdd with $\|T\|_{p_0 \to q_0} \le M_0$

Linear Interpolation Theory

Let (Σ,μ) & (Ω,ν) be two σ-finite measure spaces.

For $1 \le p_0, p_1, q_0, q_1 \le \infty$ suppose

(i) $T : L^{p_0}(\mu) \to L^{q_0}(\nu)$ linear Bdd with $\|T\|_{p_0 \to q_0} \le M_0$

(ii) $T : L^{p_1}(\mu) \to L^{q_1}(\nu)$ linear Bdd with $\|T\|_{p_1 \to q_1} \le M_1$

Linear Interpolation Theory

Theorem of Riesz-Thorin

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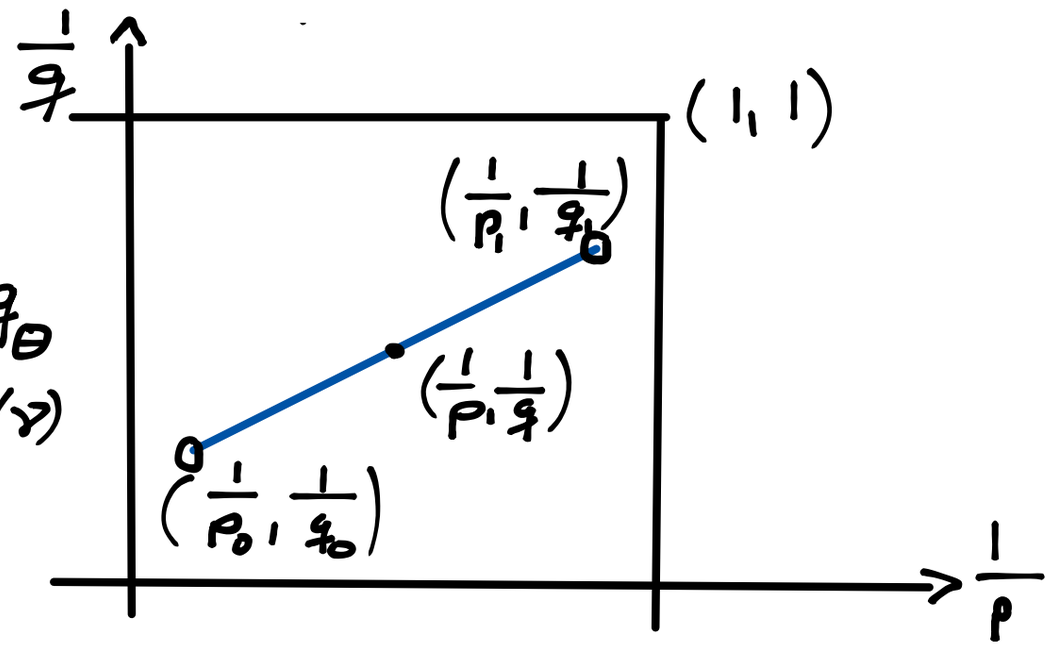
Linear Interpolation Theory

(I) T admits a unique linear Bdd extension $L^{p_\theta}_{(\mu)} \to L^{q_\theta}_{(\nu)}$

with $\|T\|_{p_\theta \to q_\theta} \leq M_0^{1-\theta} M_1^{\theta}$

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \& \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \,, \; \theta \in (0,1)$$



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(II) Let $X_0 = L^{p_0}(\mu)$ & $X_1 = L^{p_1}(\mu)$ & $X_\theta = L^{p_\theta}(\mu)$.

One has

(i) $X_0 \cap X_1 \hookrightarrow X_\theta \hookrightarrow X_0 + X_1$

(X_θ is an "intermediate space" of "interpolation couple" (X_0, X_1))

Linear Interpolation Theory

(ii) For every $T: X_0 + X_1 \to X_0 + X_1$ lin. bdd.
satisfying $T|_{X_i}: X_i \to X_i$ lin bdd., $i = 0, 1$,
one has $T|_{X_\theta}: X_\theta \to X_\theta$ lin. bdd.

(X_θ is an "interpolation space")

Linear Interpolation Theory

Remarks

- Riesz-Thorin's interpolation theorem is as the prime example of the class of complex interpolation methods.

Linear Interpolation Theory

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- Riesz-Thorin's interpolation theorem is as the prime example of the class of complex interpolation methods.
- Generalisation: Calderón's interpolation thm [Calderón, Studia Math. 24 (1964)]

Linear Interpolation — k-Methods

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Linear Interpolation — K-Methods

Let (X_0, X_1) be an interpolation couple.

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Linear Interpolation — K-Methods

Let (X_0, X_1) be an interpolation couple.

For $x \in X_0 + X_1$ & $t > 0$, one sets

$$K(t,x) := \inf_{\substack{x = x_0 + x_1 \\ x_i \in X_i}} \left(\|x_0\|_{X_0} + t \|x_1\|_{X_1} \right)$$

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Then for every $0 < \Theta < 1$, $1 \leq p \leq \infty$,

the real interpolation space

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$$(X_0, X_1)_{\Theta,p,K} \overset{\text{Def.}}{:=} \left\{ x \in X_0 + X_1 \;\middle|\; \|x\|_{\Theta,p} := \| t^{-\Theta} K(t,x) \|_{L^p_*(0,\infty)} < \infty \right\}$$

Linear Interpolation — $\mathcal{K}$-Methods

Example: For $1 \le p_0 < p_1 \le \infty$, $1 \le p \le \infty$,

$$\left(L^{p_0}(\mu), L^{p_1}(\mu)\right)_{\theta,p,k} = L^{p,q}(\mu)$$ Lorentz space

With $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Ref: Butzer, Berens (1967)
Thm. 3.3.10

Linear Interpolation — K-Methods

Interpolation theorem (K-method)

Let (X_0, X_1) & (Y_0, Y_1) be two interpolation couples,

- $T : X_0 \to Y_0$ lin. bdd.
- $T : X_1 \to Y_1$ lin. bdd.

$$\Longrightarrow \quad \|T\|_{\mathcal{L}((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)} \quad \text{for all } \begin{matrix} 0<\theta<1 \\ 1 \leq p \leq \infty \end{matrix}$$

Linear Interpolation — L-Methods

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Linear Interpolation — L-Methods

Let (X_0, X_1) be an interpolation couple.

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Linear Interpolation — L-Methods

Let (X_0, X_1) be an interpolation couple.

For $x \in X_0 + X_1$, $1 \le p_0, p_1 < \infty$ & $t > 0$, one sets

$$L(t,x) := \inf_{\substack{x = x_0 + x_1 \\ x_i \in X_i}} \left(\|x_0\|_{X_0}^{p_0} + t \|x_1\|_{X_1}^{p_1} \right)$$

Linear Interpolation — L-Methods

Let (X_0, X_1) be an interpolation couple.

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Then for every $0 < \Theta < 1$, $1 \le p \le \infty$, the interpolation space

Linear Interpolation — L-Methods

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Then for every $0 < \Theta < 1$, $1 \le p \le \infty$,
the interpolation space

$$(X_0, X_1)_{\Theta,p,L} \overset{\text{Def.}}{:=} \left\{ x \in X_0 + X_1 \;\middle|\; \|x\|_{\Theta,p,L} := \| t^{-\Theta} L(t,x) \|_{L^p_*(0,\infty)} < \infty \right\}$$

Linear Interpolation — L-Methods

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$$(X_0, X_1)_{\Theta,p,L} \overset{\text{Def.}}{:=} \left\{ x \in X_0 + X_1 \;\middle|\; \|x\|_{\Theta,p,L} := \| t^{-\Theta} L(t,x) \|_{L^p_*(0,\infty)} < \infty \right\}$$

Linear Interpolation — L-Methods

Relation to $(X_0, X_1)_{\theta,p,K}$:

One has, for all $1 \le p_0, p_1 < \infty$, $0 < \theta < 1$ & $1 \le p \le \infty$

$$(X_0, X_1)_{\theta,p,L} \overset{\text{alg.}}{=} (X_0, X_1)_{z,q}$$

$$z := \frac{1-z}{z} = \frac{1-\theta}{\theta} \cdot \frac{p_0}{p_1}$$

$$q = \left((1-\theta)p_0 + \theta p_1\right) \cdot p$$

Linear Interpolation — The mean Method

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Linear Interpolation — The mean Method

JACQUES-LOUIS LIONS

JAAK PEETRE

Sur une classe d'espaces d'interpolation

Publications mathématiques de l'I.H.É.S., tome 19 (1964), p. 5-68.

<http://www.numdam.org/item?id=PMIHES_1964__19__5_0>

Linear Interpolation — The mean Method

For given interpolation couple (X_0, X_1),

$0 < \Theta < 1$ and $1 \leq p_0, p_1 \leq \infty$

Linear Interpolation — The mean Method

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$0 < \theta < 1$ and $1 \leq p_0, p_1 \leq \infty$

the **mean space** (espaces de moyennes)

$$(X_0, X_1)_{\theta, p_0, p_1} = \{ x \in X_0 + X_1 \mid x \text{ satisfies } (*) \}$$

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Linear Interpolation — The mean Method

For given interpolation couple (X_0, X_1),

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the **mean space** (espaces de moyennes)

$$(X_0, X_1)_{\theta, p_0, p_1} = \{ x \in X_0 + X_1 \mid x \text{ satisfies } (*) \}$$

$(*)$ $\begin{cases} \exists \text{ measurable functions} \\ \quad V_0 : (0,\infty) \to X_0 \,,\, V_1 : (0,\infty) \to X_1 \\ \text{s.th. } x = V_0(t) + V_1(t) \text{ in } X_0 + X_1 \text{ for a.e. } t \in (0,1) \\ t \mapsto t^{-\theta} V_0(t) \in L^{p_0}_*(X_0) \;\&\; t \mapsto t^{-\theta} V_1(t) \in L^{p_1}_*(X_1) \end{cases}$

Linear Interpolation — The mean Method

$$(*)\quad \begin{cases} \exists \text{ measurable functions} \\ \quad V_0 : (0,\infty) \to X_0 \,,\, V_1 : (a\infty) \to X_1 \\ \text{s.th. } X = V_0(t) + V_1(t) \text{ in } X_0 + X_1 \text{ for a.e. } t \in (0,1) \\ t \mapsto t^{-\theta} v_0(t) \in L^{p_0}_*(X_0) \;\&\; t \mapsto t^{-\theta} V_1(t) \in L^{p_1}_*(X_1) \end{cases}$$

$$\|x\|_{\theta,p_0,p_1} := \inf_{\substack{x = V_0(t)+V_1(t) \\ V_0,V_1 \text{ satisfy } (*)}} \max\left\{ \|t^{-\theta} v_0\|_{L^{p_0}_*(X_0)}, \|t^{1-\theta} v_1\|_{L^{p_1}_*(X_1)} \right\}$$

Linear Interpolation — The mean Method

Interpolation theorem (Mean-method)

Let (X_0, X_1) & (Y_0, Y_1) be two interp couples

& $T : X_0 + X_1 \longrightarrow Y_0 + Y_1$ be a linear mapping

s.th. $T_{|X_0} : X_0 \longrightarrow Y_0$ is $\|\cdot\| \leq M_0$

$T_{|X_1} : X_1 \longrightarrow Y_1$ is $\|\cdot\| \leq M_1$

Linear Interpolation — The mean Method

Interpolation theorem (Mean-method)

Then, for every $1 \le p_0 \le p_1 \le \infty$ & $0 < \theta < 1$,

$$T : (X_{01}, X_1)_{\theta,p_0,p_1} \longrightarrow (Y_{01}, Y_1)_{\theta,p_0,p_1}$$

$$T|_{(X_{01},X_1)_{\theta,p_0,p_1}} \text{ is bdd} \le M_0^{1-\theta} \cdot M_1^{\theta}.$$

Linear Interpolation — The mean Method

Relation to $(X_0, X_1)_{\theta,P,K}$:

Sur le nombre de paramètres dans la définition de certains espaces d'interpolation

Nota di JAAK PEETRE (a Lund)

Ricerche Mat. 12 (1963), 248–261.

Linear Interpolation — The mean Method

Relation to $(X_0, X_1)_{\theta,p,K}$:

3. Démonstration que les espaces de moyennes ne dépendent que de θ et de p.

Le but de ce N° est de démontrer del théorème suivant, ce qui est la contribution essentielle de cet article.

THÉORÈME 1. *Quels que soient* θ, p_0, p_1 *on a :*

$$(1) \qquad (A_0, A_1)_{\theta,p_0,p_1} = (A_0, A_1)_{\theta,p,p} \text{ pour } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

les injections étant continues ; autrement dit, les espaces $(A_0, A_1)_{\theta,p_0,p_1}$, *où interviennent apparemment trois paramètres :* θ, p_0, p_1, *ne dépendent effectivement que de deux paramètres :* θ, p.

Nonlinear Interpolation

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Nonlinear Interpolation

(1) Marcinkiewicz interpolation theorem

J. Marcinkiewicz, *Sur l'interpolation d'opérations*. C.R. Acad. Sci. Paris **208**, 1272–1273 (1939).

Nonlinear Interpolation

(1) Marcinkiewicz interpolation theorem

J. Marcinkiewicz, *Sur l'interpolation d'opérations*. C.R. Acad. Sci. Paris **208**, 1272–1273 (1939).

is stated for quasi-subadditive operators T,

$$|T(x+y)| \leq C\,(|Tx| + |Ty|)$$

Nonlinear Interpolation

(II) Gagliardo's nonlinear interpolation theorem.

E. Gagliardo, *Interpolations d'espaces de Banach et applications*, **II, III,** C.R. Acad. Sci. Paris **248,** 3388–3390, 3517–3518 (1959).

Nonlinear Interpolation

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(III) Orlicz (interpolation of Lipschitz mappings)

W. Orlicz, *On a class of operators over the space of integrable functions*. Studia Math. **14,** 302–309 (1954).

Nonlinear Interpolation

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(III) Orlicz (interpolation of Lipschitz mappings)

W. Orlicz, *On a class of operators over the space of integrable functions.* Studia Math. **14,** 302–309 (1954).

(IV) Shimogaki

T. Shimogaki, *An interpolation theorem on Banach function spaces.* Studia Math. **31,** 233–240 (1968).

G. G. Lorentz-T. Shimogaki, *Interpolation theorems for operators in function spaces.* J. Functional Analysis **2,** 31–51 (1968).

Nonlinear Interpolation

(v) Peetre

Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) **12(35)** (1970),

Nonlinear Interpolation

(V) Peetre

Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) **12(35)** (1970),

(VI) Tartar

Nonlinear Interpolation

(V) Peetre

Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) **12(35)** (1970),

(VI) Tartar

JOURNAL OF FUNCTIONAL ANALYSIS 9, 469–489 (1972)

Interpolation non Linéaire et Régularité

L. TARTAR

11 rue Servandoni, 75 Paris 6e, France

Communicated by J. L. Lions

Nonlinear Interpolation

(VI) Tartar

THÉORÈME 4. *On suppose qu'on a* (11) *avec l'hypothèse supplémentaire* $\exists \alpha, \beta : 0 < \alpha \leqslant 1 ; 0 < \beta$ tels que

$$\| Ta - Tb \|_{B_0} \leqslant c_0 \| a - b \|^{\beta}_{A_0}, \qquad \forall a, b \in A_0, \tag{11bis}$$

$$\| Ta - Tb \|_{B_1} \leqslant c_1 \| a - b \|^{\alpha}_{A_1}, \qquad \forall a, b \in A_0 + A_1 : a - b \in A_1 .$$

Alors si $0 < \theta < 1$, $1 \leqslant p < +\infty$ *on a*

$$\| Ta - Tb \|_{(B_0, B_1)_{\eta, q}} \leqslant c \cdot C_0^{1-\eta} C_1^{\eta} \| a - b \|^{(1-\eta)\beta + \eta\alpha}_{(A_0, A_1)_{\theta, p}} \qquad \forall a, b \in (A_0, A_1)_{\theta, p} . \tag{13}$$

$$\frac{1-\eta}{\eta} = \frac{1-\theta}{\theta} \frac{\alpha}{\beta}, \qquad q = \max \left(1, \frac{p}{(1-\eta)\beta + \eta\alpha} \right)$$

Theorem (Coulhon, H. '15)

Let (X_0, X_1) & (Y_0, Y_1) be two interpolation couples.
Suppose $\exists\, 0 < \alpha_0, \alpha_1 < \infty$ & $M_0, M_1 \geq 0$ & $T : X_0 + X_1 \to Y_0 + Y_1$ be a mapping with $X_0 \cap X_1 \subseteq D(T)$ &

$$\|Tu - T\hat{u}\|_{Y_0} \leq M_0 \|u - \hat{u}\|_{X_0}^{\alpha_0}$$

$$\|Tu - T\hat{u}\|_{Y_1} \leq M_1 \|u - \hat{u}\|_{Y_0}^{\alpha_1}$$

for all $u, \hat{u} \in X_0 \cap X_1$.

Then for every $0 < \theta < 1$, $1 \leq q_0, q_1 \leq \infty$ (excluding $q_0 = q_1 = \infty$) satisfying $q_0 \geq \frac{1}{\alpha_0}$, $q_1 \geq \frac{1}{\alpha_1}$, one has

$$\|Tu - T\hat{u}\|_{(Y_0, Y_1)_{\theta, q_0, q_1}} \leq \left(\frac{2\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta} M_1^{\theta} C^{\alpha}_{r_0 r_1} \|u - \hat{u}\|_{(X_0, X_1)_{\gamma, s_0, s_1}}$$

for every $u, \hat{u} \in X_0 \cap X_1$,

where $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\rho_0 = \alpha_0 q_0$, $\gamma = \frac{\theta \alpha_1}{(1-\theta)\alpha_0 + \theta \alpha_1}$

& $1 \leq s_i \leq \rho_i$ for $i = 0, 1$ $\rho_1 = \alpha_1 q_1$ $\alpha = (1-\theta)\alpha_0 + \theta\alpha$.

Idea of the proof:

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- Let $x, \hat{x} \in X_0 \cap X_1$. $\hookrightarrow (X_0, X_1)_{\mathcal{Z}, P_0, P_1}$

Idea of the proof:

- Let $u, \hat{u} \in X_0 \cap X_1$. $\hookrightarrow (X_0, X_1)_{\frac{1}{2}, P_0, P_1}$

$\Rightarrow \exists\ (v_0, v_1)$ s.th. $v_i : (0, \infty) \to X_i$ is measurable, $u = v_0(t) + v_1(t)$ a.e. t & $t^{-z} v_0(t) \in L^{P_0}_*(X_0)$, $t^{1-z} v_1(t) \in L^{P_1}_*(X_1)$.

Idea of the proof:

▷ Let $u, \hat{u} \in X_0 \cap X_1$. $\hookrightarrow (X_0, X_1)_{\theta, P_0, P_1}$

$\Rightarrow \exists\ (v_0, v_1)$ s.th. $v_i : (0,\infty) \to X_i$ is measurable, $u = v_0(t) + v_1(t)$ a.e. t & $t^{-\theta} v_0(t) \in L^{p_0}_*(X_0)$, $t^{1-\theta} v_1(t) \in L^{p_1}_*(X_1)$.

▷ Also $u = v_0(t^\lambda) + v_1(t^\lambda)$ for a.e. $t > 0$ & $\forall\ \lambda > 0$

Idea of the proof:

▷ Let $u, \hat{u} \in X_0 \cap X_1$. $\hookrightarrow (X_0, X_1)_{\theta, P_0, P_1}$

$\Rightarrow \exists\ (v_0, v_1)$ s.th. $v_i : (0,\infty) \to X_i$ is measurable, $u = v_0(t) + v_1(t)$ a.e. t & $t^{-\theta} v_0(t) \in L^{p_0}_*(X_0)$, $t^{1-\theta} v_1(t) \in L^{p_1}_*(X_1)$.

▷ Also $u = v_0(t^\lambda) + v_1(t^\lambda)$ for a.e. $t > 0$ & $\forall\ \lambda > 0$ Take $\lambda = \frac{\theta}{2\alpha_0}$

Idea of the proof:

Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta, q_0, q_1}$

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Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta, q_0, q_1}$

Set $W_0(t) := T(V_0(t^{-1})+\hat{u}) - Tu$

$W_1(t) = T(u+\hat{u}) - T\hat{u} - W_0(t)$

Idea of the proof:

Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta|q_0|q_1}$

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$W_1(t) = T(u+\hat{u}) - T\hat{u} - W_0(t)$

$\Rightarrow W_2 : (0,\infty) \to Y_2$ is measurable

Idea of the proof:

Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta|q_0|q_1}$

Set $W_0(t) := T(v_0(t^\lambda) + \hat{u}) - Tu$

$W_1(t) = T(u+\hat{u}) - T\hat{u} - W_0(t)$

$\Rightarrow W_2 : (0,\infty) \to Y_2$ is measurable

Sub. $s = t^\lambda$

$$\Longrightarrow \|t^{-\theta} W_0\|_{L^{q_0}_*(Y_0)} \leq M_0 \left(\frac{2^{\alpha_0}}{\theta}\right)^{\frac{1}{q_0}} \|s^{\frac{\alpha_0}{\lambda}} v_0\|_{L^{p_0}_*(X_0)}$$

Idea of the proof:

Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta|q_0|q_1}$

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Sub. $s = t^\lambda$

$$\Longrightarrow \|t^{-\theta} W_0\|_{L^{q_0}_*(Y_0)} \leq M_0 \left(\frac{2\alpha_0}{\theta}\right)^{\frac{1}{q_0}} \|s^{-z} V_0\|^{\alpha_0}_{L^{p_0}_*(X_0)}$$

$\lambda = \frac{1-\theta}{(1-z)\alpha_1}$

$$\Longrightarrow \|t^{1-\theta} W_1\|_{L^{q_1}_*(Y_1)} \leq M_1 \left(\frac{2\alpha_0}{\theta}\right)^{\frac{1}{q_1}} \|s^{1-z} V_1\|^{\alpha_1}_{L^{p_1}_*(X_1)}$$

Idea of the proof:

Aim: $T(u+\hat{u}) - T\hat{u} \in (Y_0, Y_1)_{\theta|q_0|q_1}$

Set $W_0(t) := T(v_0(t^\lambda) + \hat{u}) - Tu$

$W_1(t) = T(u+\hat{u}) - T\hat{u} - W_0(t)$

$\Rightarrow W_2 : (0,\infty) \to Y_2$ is measurable

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$$\Longrightarrow \|t^{-\theta} W_0\|_{L^{q_0}_*(X_0)} \le M_0 \left(\frac{2\alpha_0}{\theta}\right)^{\frac{1}{q_0}} \|s^{-z} v_0\|^{\alpha_0}_{L^{p_0}_*(X_0)}$$

$\lambda = \frac{1-\theta}{(1-z)\alpha_1}$

$$\Longrightarrow \|t^{1-\theta} W_1\|_{L^{q_1}_*(X_1)} \le M_1 \left(\frac{2\alpha_0}{\theta}\right)^{\frac{1}{q_1}} \|s^{1-z} v_1\|^{\alpha_1}_{L^{p_1}_*(X_1)}$$

Combining & $\inf_{v_0, v_1}$ & $u \mapsto u - \hat{u}$. $\square$

Application to nonlinear semigroups

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p-Laplace

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Δ_p: p-Laplace

Our interest is in:

$$u(0) \in L^q, \quad 1 \le q < \infty \quad \Longrightarrow \quad \|u(t)\|_\infty \lesssim e^{\omega t \beta_q} t^{-\alpha_q} \|u(0)\|_q \quad \forall\, t > 0 \,.$$

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- $u(t,\cdot) \in L^\infty \Rightarrow \frac{\partial u}{\partial t} \in L^1_{loc}$

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- $u(t_1) \in L^\infty \Rightarrow \frac{\partial u}{\partial t} \in L^1_{loc}$

In particular:

> Entropy/mild solutions are strong.

The main idea to obtain L^{p_0}–L^∞ estimates

The main idea to obtain $L^{\frac{n}{2}}$–L^∞ estimates

- Construction of a one-parameter family of Sobolev-inequalities:

$$\|u\|_{K_q}^q \leq C_q \cdot \langle Au, |u|^{q-p}u \rangle \quad \text{for all } q \geq p$$

For $q \geq p$

$$\langle -\Delta_p u, |u|^{q-p} u \rangle$$

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$$\text{for } q \geq p$$

$$\langle -\Delta_p u, |u|^{q-p} u \rangle = (q-p+1) \int_\Omega |\nabla u|^p \, |u|^{q-p} \, dx$$

For $q \geq p$

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$$= (q-p+1) \int_\Omega \left| \nabla u \cdot |u|^{\frac{q-p}{p}} \right|^p dx$$

$$\text{for } q \geq p$$

$$\left\langle -\Delta_p u, |u|^{q-p} u \right\rangle = (q-p+1) \int_{\Omega} |\nabla u|^p \, |u|^{q-p} \, dx =$$

$$\left(\frac{q-p}{p}+1\right)^p \int_{\Omega} |\nabla (|u|^{\frac{q-p}{p}} u)|^p \, dx$$

$$= \frac{q-p+1}{(q-p+1)^p} \int_{\Omega} |\nabla (|u|^{\frac{q-p}{p}} u)|^p \, dx =$$

$$= (q-p+1) \int_{\Omega} |\nabla u \cdot |u|^{\frac{q-p}{p}}|^p \, dx$$

$$p^* = \frac{pd}{d-p}$$

$$\geq c_{qp} \left(\int_\Omega |u| \frac{q-p}{p} |u|^{p^*} dx \right)^{\frac{p}{p^*}}$$

$$\frac{\left(\frac{q-p}{p}+1\right)^p}{\left(q-p+1\right)} \int_\Omega |\nabla(|u|^{\frac{q-p}{p}} u)|^p dx \int_\Omega |u|^{q-p} dx$$

$$= \frac{p^p}{(q-p+1)^p} \int_\Omega |\nabla u \cdot |u|^{\frac{q-p}{p}} u|^p dx \int_\Omega |u|^{q-p} dx$$

$$= (q-p+1) \int_\Omega |\nabla u|^p |u|^{q-p} dx = (q-p+1) \left\langle -\Delta_p u, |u|^{q-p} u \right\rangle$$

für $q \geq p$

For $q \geq p$

$$\langle -\Delta_p u, |u|^{q-p} u \rangle = (1+d-p) \int_\Omega |\nabla u|^p |u|^{q-p} dx$$

$$= (q-p+1) \int_\Omega |\nabla u \cdot |u|^{\frac{d-p}{p}}|^p dx$$

$$= \frac{(q-p+1)}{\left(\frac{q-p}{p}+1\right)^p} \int_\Omega |\nabla (|u|^{\frac{q}{p}} u)|^p dx$$

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$$\geq C_{qp} \left(\int_\Omega | |u|^{\frac{q-p}{p^*} u} |^{p^*} dx \right)^{\frac{p}{p^*}}$$

& $\kappa := \frac{d}{d-p} > 1$

$$= C_{qp} \left(\int_\Omega |u|^{\frac{qd}{d-p}} dx \right)^{\frac{d-p}{d}} = C_{qp} \|u\|_{q\kappa}^q$$

The main idea to obtain $L^{\frac{n}{2}}$–L^∞ estimates

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- For every $q > p$, one deduces

$$\|u(t)\|_{K_q} \leq C_q^* \, t^{-\frac{1}{q}} \, \|u\|_{q-p+2}^{\frac{q-p+2}{q}} \quad \text{for all } t > 0$$

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- Parabolic Moser iteration

What is known about the regularisation effect of non local diffusion equations?

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1st Example

$$\partial_t u - (-\Delta_p)^s u + \beta(u) = 0 \quad \text{in } \Omega \times (0,\infty)$$

where $\langle -(-\Delta_p)^s u, v\rangle = \frac{1}{2}\int_\Omega \int_\Omega \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))|v(y)-v(x)}{|y-x|^{d+sp}}\,dx\,dy$

for $1<p<\infty$ & $0<s<1$.

"fractional p-Laplace operator" (weak)

2nd Example

$$(1)\quad \begin{cases} \partial_t u + |\nabla u|^{p-2}\langle \nabla u, \vec{\nu}\rangle + f(u) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ -\Delta_p u = 0 & \text{in } \Omega \times (0,\infty) \end{cases}$$

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If $P: \varphi \mapsto P\varphi := u$ unique solution of

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If $P: \varphi \mapsto P\varphi := u$ unique solution of

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then $\Lambda : \varphi \mapsto |\nabla P\varphi|^{p-2}\langle \nabla P\varphi, \vec{\nu}\rangle\big|_{\partial\Omega}$

is the Dirichlet-to-Neumann operator.

2nd Example

$$(1)\quad \begin{cases} \partial_t u + |\nabla u|^{p-2}\langle \nabla u, \vec{\nu}\rangle + f(u) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ -\Delta_p u = 0 & \text{in } \Omega \times (0,\infty) \end{cases}$$

$\Longrightarrow$ One can rewrite (1) as

$$\partial_t \varphi + \Lambda \varphi + f(\varphi) = 0 \quad \text{on } \partial\Omega \times (0,\infty)$$

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The construction of a one-parameter family of Sobolev inequalities

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The constraction of a one-parameter family of Sobolev inequalities

$$\langle \mathcal{L}\varphi, |\varphi|^{q-p}\varphi \rangle = \int_\Omega |\nabla^p \varphi|^{p-2} \nabla^p \varphi \cdot \nabla^p (|\varphi|^{q-p}\varphi)\, dx$$

2nd Example

The constraction of a one-parameter family of Sobolev inequalities

$$\langle \mathcal{A}\varphi, |\varphi|^{q-p}\varphi \rangle = \int_\Omega |\nabla_p \varphi|^{p-2} \nabla_p \varphi \cdot \nabla_p (|\varphi|^{q-p}\varphi)\, dx$$

BUT $\quad \nabla_p(|\varphi|^{q-p}\varphi) \neq |\nabla_p \varphi|^{q-p} \nabla_p \varphi$

does not work!

2 Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^q$-L^∞-reg.

Let $A+\omega I$ be m-completely accretive in $L^q(\Sigma,\mu)$ with dense domain & $0\in A0$ & $\omega\in\mathbb{R}_+$.

If there are $1\le q,r<\infty$, $\sigma>0$ & $C>0$ s.th. $\frac{r}{\sigma}>1$ &

(Sob) $$\|u-\hat u\|_r^\sigma \le C\left([u-\hat u, v-\hat v]_q + \omega\|u-\hat u\|_q^q\right)$$

for all $(u,v),(\hat u,\hat v)\in A$, then $\{T_t\}_{t\ge 0}\sim -A$ satisfies

$$\|T_t u - T_t\hat u\|_\infty \lesssim t^{-\delta} e^{\omega t(\beta^*+1)}\|u-\hat u\|_{\frac{r}{\sigma}m_0}^{\gamma} \quad \forall u,\hat u\in L^{\frac{r}{\sigma}m_0},$$

for any $m_0>\sigma$ satisfying $\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q>0$

with $\delta=\frac{1}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q}$, $\beta^*=\frac{\left(\frac{q}{\sigma}+1\right)\frac{r}{q}-1}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q}$, $\gamma=\frac{\left(\frac{r}{\sigma}-1\right)m_0}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q}$.

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One has

- $\| T_t u - T_t \hat{u} \|_r \leq \left(\frac{c}{q}\right)^{\frac{1}{b}} e^{\omega t \left(\frac{a}{b}+1\right)} t^{-\frac{1}{b}} \|u - \hat{u}\|_q^{\frac{a}{b}}$

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A new "nonlinear interpolation theorem"

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A new "nonlinear interpolation theorem"

$\Longrightarrow$ for every $0<\theta<1$ there is $\eta(\theta) \in (0,1)$ s.th.

$$\|T_t u - T_t \hat{u}\|_{\frac{r}{1-\theta}} \leq \left[\frac{q/b}{(1-\theta)\,q/b+\theta}\right]^{\frac{1-\theta}{r}} \times \left[C \cdot e^{\omega t} \tilde{\beta}\, t^{-b}\right]^{1-\theta} \times$$

$$\times\, C^{*(1-\theta)\frac{q}{b}+\theta}_{\infty,q} \times \|u-\hat{u}\|_q^{\frac{(1-\theta)\frac{q}{b}+\theta}{1-\eta(\theta)}}$$

Idea of the Proof of Theorem 2:

With $K = \frac{\tau}{\sigma} > 1$

take $\theta_m = 1 - \frac{1}{m} \cdot \frac{\tau}{K}$ $\forall m \geq m_0$ s.t. $\frac{q}{1-z(\theta_m)} \geq 1$

$\square$

Thank You!

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