

Fundamental gap conjecture

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SYDNEY

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TU Dresden

This talk is based on the 2 papers:

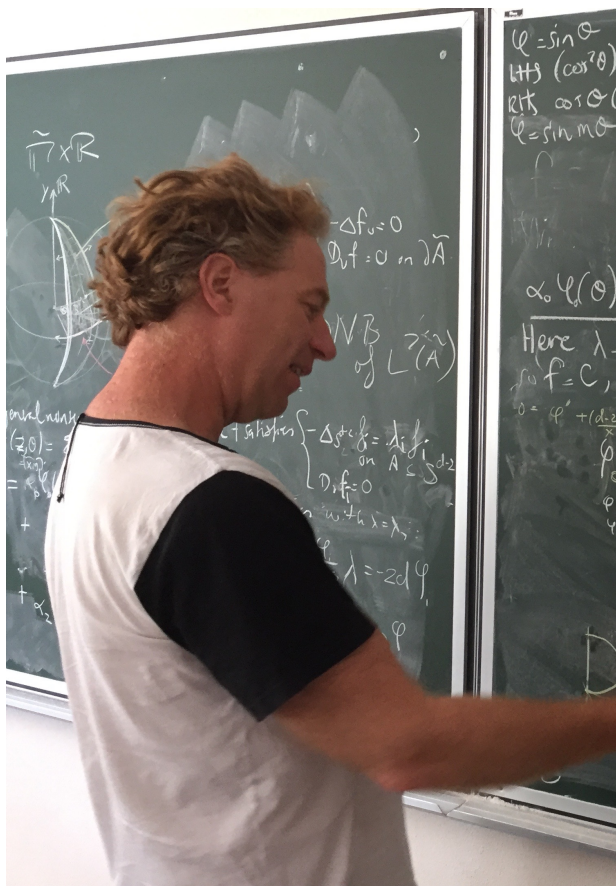
1. Camb. J. Math., vol. 8 (2020), no 2.
2. Proc. Amer. Math. Soc., vol. 149 (2021)

Preprints are available at:

www.maths.usyd.edu.au/u/dhauer



Joint work with



Ben Andrews



Julie Clutterback

Fundamental Gap Conjecture



Fundamental Gap Conjecture

In 1983, Michiel van den Berg [5] stated

"for a bounded convex domain $\Omega \subseteq \mathbb{R}^d, d \geq 1$,
& for a convex potential $V: \Omega \rightarrow \mathbb{R}$,
the difference between the 1st & 2nd Dirichlet-eigen-
values $\lambda_{\infty 1}^V, \lambda_{\infty 2}^V$ of the Schrödinger operator $-\Delta + V$

satisfy

$$\Gamma(V) := \lambda_{\infty 2}^V - \lambda_{\infty 1}^V \geq \frac{3\pi^2}{D},$$

where $D := \text{diam}(\Omega)$."



Fundamental Gap Conjecture

The difference $\lambda_{\infty 2}^V - \lambda_{\infty 1}^V$ is called
fundamental gap or spectral gap.



Fundamental Gap Conjecture

In quantum physics, the Schrödinger (eigenvalue) eq.

$$-\frac{\hbar^2}{2m} \Delta \varphi - \frac{e^2}{r(x)} \varphi = E \varphi$$

models the motion of an electron in an electric field generated by a nucleus with exactly one proton.



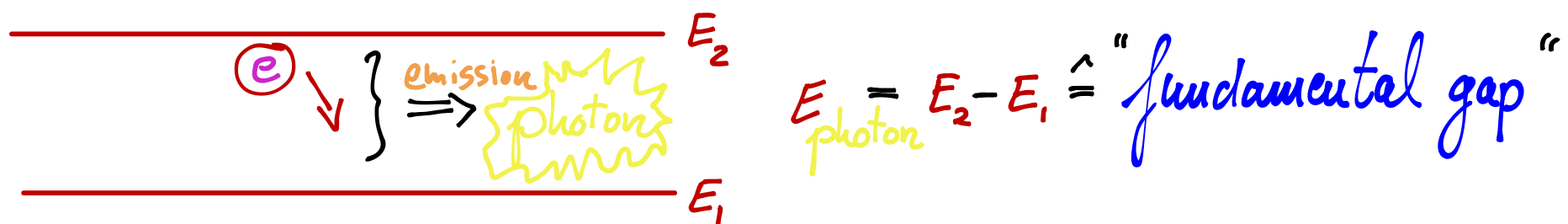
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In quantum physics, the Schrödinger (eigenvalue) eq.

$$-\frac{\hbar^2}{2m} \Delta \psi - \frac{e^2}{r(x)} \psi = E \psi$$

models the motion of an electron in an electric field generated by a nucleus with exactly one proton.

As an electron changes the energy levels it emits a photon.



Fundamental Gap Conjecture

The *fundamental gap* appears naturally

- in the convergence rate by approximating ground & excited state energies on *quantum computers*

(see, e.g., Hadfield & Papageorgiou [Quantum Inf. Proc. 14 (2014)])

- in numerical methods for large eigenvalue problems (Saad [SIAM vol. 66, 2011])



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$$-\varphi''_{\infty n} = \lambda_{\infty n} \varphi_{\infty n} \quad \text{on } \Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)$$



The case $d=1$ and $V \equiv 0$

$$-\varphi''_{\infty n} = \lambda_{\infty n} \varphi_{\infty n} \quad \text{on } \Omega = \left(-\frac{D}{2}, \frac{D}{2}\right)$$

$$\Rightarrow \lambda_{\infty n} = \frac{h^2 \pi^2}{D^2} \quad \Rightarrow \lambda_{\infty 2} - \lambda_{\infty 1} = \frac{3\pi^2}{D^2}$$



Fundamental Gap Conjecture

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The fundamental gap conjecture was also independently suggested by

= Ashbaugh & Benguria [2] in 1989



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= Yau [10] in 1986.



Fundamental Gap Conjecture

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- Horváth [7] partly removed the above symmetry assumption by allowing single-well potentials V with minimum at $x=0$.
- In 1994, Lavine [8] proved the fundamental gap conjecture in dimension $d=1$.



Fundamental Gap Conjecture

In higher dimensions

□ Bañuelos & Méndez-Hernández[4] proved the fundamental gap conjecture for $V \equiv 0$.



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Nonoptimal bounds

- = In 1985, Singer, Wong, Yau & Yau [9] proved $\lambda_{\infty 2}^V - \lambda_{\infty 1}^V \geq \frac{\pi^2}{4D^2}$



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- In 1985, Singer, Wong, Yau & Yau [9] proved $\lambda_{\infty 2}^V - \lambda_{\infty 1}^V \geq \frac{\pi^2}{4D^2}$
- This result was improved by Yu & Zhong [11] to

$$\lambda_{\infty 2}^V - \lambda_{\infty 1}^V \geq \frac{\pi^2}{D^2}.$$

Fundamental Gap Conjecture

In 2011, the fundamental gap conjecture in the
in the Dirichlet case was completely solved by
Andrews & Clutterbuck [1]





Problem 1

Does the fundamental gap conjecture also hold for *other* boundary conditions?



The Classical Eigenvalue Problem

For $0 \leq \alpha \leq +\infty$, Ω a bounded domain in $\mathbb{R}^d$, $V \in L^\infty(\Omega)$,
the classical eigenvalue problem for the Schrödinger
operator $-\Delta + V$ on Ω is given by



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$$(E_\alpha) \quad \begin{cases} -\Delta \varphi_\alpha + V \varphi_\alpha = \lambda_\alpha^V \varphi_\alpha & \text{in } \Omega \\ D_\alpha \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$



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$$(E_\alpha) \quad \begin{cases} -\Delta \varphi_\alpha + V \varphi_\alpha = \lambda_\alpha \varphi_\alpha & \text{in } \Omega \\ \mathbb{D}_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$

Here, ν is the outward pointing unit normal to Ω ,
 $\mathbb{D}_\nu$ the directional derivative in direction ν &
the boundary $\partial\Omega$ is assumed to be Lipschitz.



Neumann, Robin & Dirichlet

The boundary condition $D_{\nu} \varphi_{\alpha} + \alpha \varphi_{\alpha} = 0$ on $\partial\Omega$ reduces to

Neumann

$$\alpha = 0$$

$$D_{\nu} \varphi_0 = 0$$

Robin

$$0 < \alpha < \infty$$

$$D_{\nu} \varphi_{\alpha} + \alpha \varphi_{\alpha} = 0$$

Dirichlet

$$\text{"}\alpha = +\infty\text{"}$$

$$\varphi_{\infty} = 0$$

The eigenvalue λ_{α}^{ν} is monotone in $\alpha \nearrow$
 $\Rightarrow 0 = \lambda_0^{\nu} < \lambda_{\alpha}^{\nu} < \lambda_{\infty}^{\nu} < \infty.$

First Results in this Direction



First Results in this Direction

1. Theorem [Andrews, Clutterbuck, H.'20]

Let $\lambda_{\alpha_1}^V$ & $\lambda_{\alpha_2}^V$ be the Robin-eigenvalues on a bounded convex domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, with diameter D & $V \in L^1(\Omega)$ & $V \geq -C$, $C > 0$.

If $\varphi_{\alpha_1}^V$ is log-concave, then $\lambda_{\alpha_2}^V - \lambda_{\alpha_1}^V \geq \frac{\pi^2}{D^2}$.



Reminders



Reminders

let Ω be convex.



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Definition.

- A function $\varphi: \Omega \rightarrow \mathbb{R}_+ = (0, \infty)$ is called *log-concave* if $v(x) := \log \varphi(x)$ is *concave*.

$$\Leftrightarrow \text{for all } x, \hat{x} \in \Omega \text{ \& } \lambda \in (0, 1), \text{ one has that}$$
$$\varphi(\lambda x + (1-\lambda)\hat{x}) \geq \varphi^\lambda(x) \cdot \varphi^{1-\lambda}(\hat{x})$$

Proof of Theorem 1.



Proof of Theorem 1.

The function

$$V(x, t) := \frac{e^{-\lambda_{\alpha_2}^V t} \varphi_{\alpha_2}^V(x)}{e^{-\lambda_{\alpha_1}^V t} \varphi_{\alpha_1}^V(x)}$$



Proof of Theorem 1.

The function $V(x, t) := \frac{e^{-\lambda_2^V t} \phi_{\alpha_2}^V(x)}{e^{-\lambda_1^V t} \phi_{\alpha_1}^V(x)}$

solves $\partial_t V = \Delta V + 2 \nabla \log \phi_{\alpha_1}^V \cdot \nabla V$ on $\Omega \times (0, \infty)$



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and on $\partial \Omega \times (0, \infty)$:

$$D_\gamma V = \frac{e^{-\lambda_2^V t}}{e^{-\lambda_1^V t}} \cdot \frac{D_\gamma \varphi_{\alpha_2}^V \cdot \varphi_{\alpha_1}^V - \varphi_{\alpha_2}^V \cdot D_\gamma \varphi_{\alpha_1}^V}{(\varphi_{\alpha_1}^V)^2}$$

Proof of Theorem 1.

The function $V(x,t) := \frac{e^{-\lambda_{\alpha_2}^V t} \varphi_{\alpha_2}^V(x)}{e^{-\lambda_{\alpha_1}^V t} \varphi_{\alpha_1}^V(x)}$

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and on $\partial \Omega \times (0, \infty)$:

$$\begin{aligned} D_\gamma V &= \frac{e^{-\lambda_{\alpha_2}^V t}}{e^{-\lambda_{\alpha_1}^V t}} \cdot \frac{D_\gamma \varphi_{\alpha_2}^V \cdot \varphi_{\alpha_1}^V - \varphi_{\alpha_2}^V \cdot D_\gamma \varphi_{\alpha_1}^V}{(\varphi_{\alpha_1}^V)^2} \\ &= \frac{e^{-\lambda_{\alpha_2}^V t}}{e^{-\lambda_{\alpha_1}^V t}} \cdot \frac{-\alpha \varphi_{\alpha_2}^V \cdot \varphi_{\alpha_1}^V + \alpha \varphi_{\alpha_2}^V \cdot \varphi_{\alpha_1}^V}{(\varphi_{\alpha_1}^V)^2} \end{aligned}$$

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Proof of Theorem 1.

By hypothesis, $\log \ell_{\alpha_1}^V$ is concave

$\Leftrightarrow$ the vector field $X := 2 \nabla \log \ell_{\alpha_1}^V$ satisfies

$$\langle X(y) - X(x), y - x \rangle_{\mathbb{R}^d} \leq 0$$



Proof of Theorem 1.

By hypothesis, $\log \varphi^{\mathbf{v}}$ is concave

$\Leftrightarrow$ the vector field $X := 2 \nabla \log \varphi^{\mathbf{v}}$ satisfies

$$\langle X(y) - X(x), y - x \rangle_{\mathbb{R}^d} \leq 0$$

Definition.

A vector field X on Ω is said to have a modulus of contraction ω if there is a function $\omega: [0, D/2] \rightarrow \mathbb{R}_+$ such that
($D := \text{diam } \Omega$)

$$\langle X(y) - X(x), \frac{y-x}{|y-x|} \rangle_{\mathbb{R}^d} \leq 2\omega\left(\frac{|y-x|}{2}\right) \text{ for every } x, y \in \Omega.$$



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Thm 2.1 in [1]
 $\Rightarrow$

$$\varphi(s, t) := e^{-t \frac{\pi^2}{D^2}} \cdot \sin\left(\frac{\pi}{D} s\right)$$

for $s \in [0, D/2], t \geq 0$

is a modulus of continuity of V :



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$$V(y, t) - V(x, t) \leq 2 \varphi\left(\frac{|y-x|}{2}, t\right) \text{ for all } x, y \in \overline{\Omega} \text{ \& } t \geq 0.$$

Proof of Theorem 1.

$$\Rightarrow e^{-(\lambda_{\alpha_2}^V - \lambda_{\alpha_1}^V)t} \operatorname{osc} \frac{\varphi_{\alpha_2}}{\varphi_{\alpha_1}} \leq C \cdot e^{-t \frac{\pi^2}{D^2}} \text{ for all } t \geq 0$$



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$$\Rightarrow \lambda_{\alpha_2}^V - \lambda_{\alpha_1}^V \geq \frac{\pi^2}{D^2} \quad \square$$

1. Theorem [Andrews, Clutterbuck, H. '20]

Let $\lambda_{\alpha_1}^V$ & $\lambda_{\alpha_2}^V$ be the Robin - eigenvalues on a bounded convex domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, with diameter D & $V \in L^1(\Omega)$ & $V \geq -C$, $C > 0$.

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This result gives only a nonoptimal lower bound!



Problem 2

Is the 1st Robin - eigenfunction $\varphi_{\alpha_1}^V$ of
the Schrödinger operator $-\Delta + V$ on Ω
log-concave?



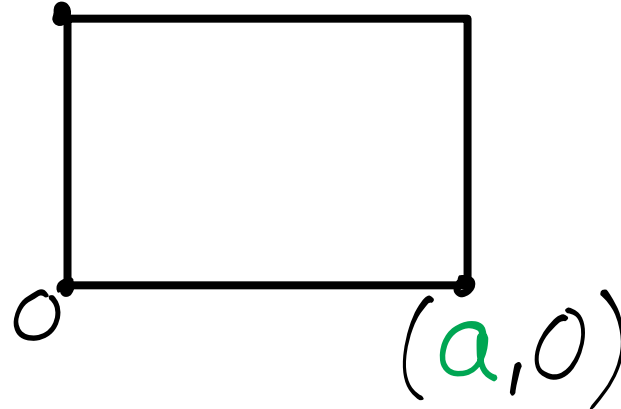


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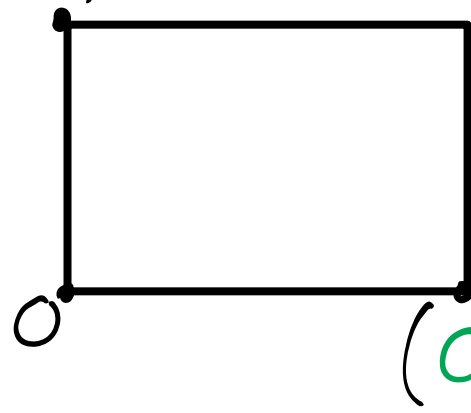
1st Example

let $\Omega =$  in $\mathbb{R}^2$



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1st Example

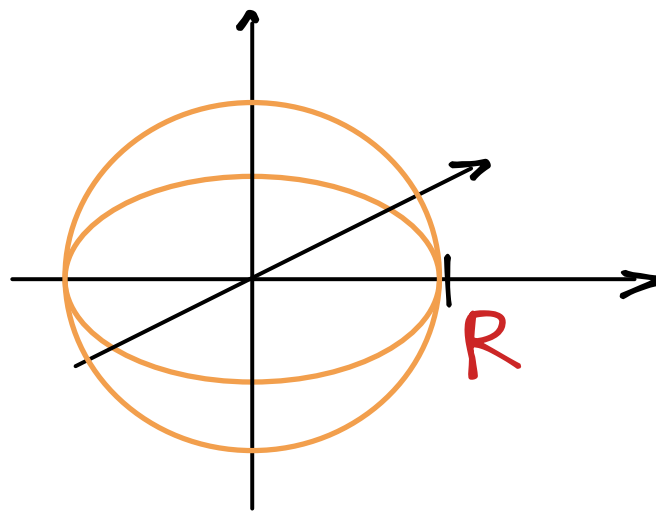
Let $\Omega =$  in $\mathbb{R}^2$

If φ_α is the 1st Robin eigenfunction on $I := (0, a) \subseteq \mathbb{R}$
& $\hat{\varphi}_\alpha$ is the 1st Robin eigenfunction on $I := (0, b) \subseteq \mathbb{R}$

then $\varphi(x, y) := \varphi_\alpha(x) \cdot \hat{\varphi}_\alpha(y)$ for $(x, y) \in \Omega$ is the 1st Robin eigenfunction on Ω & φ is log-concave ✓

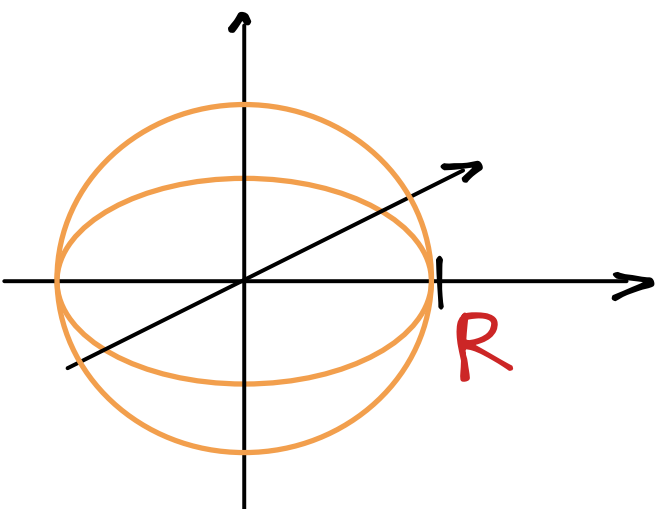
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Let $\Omega =$



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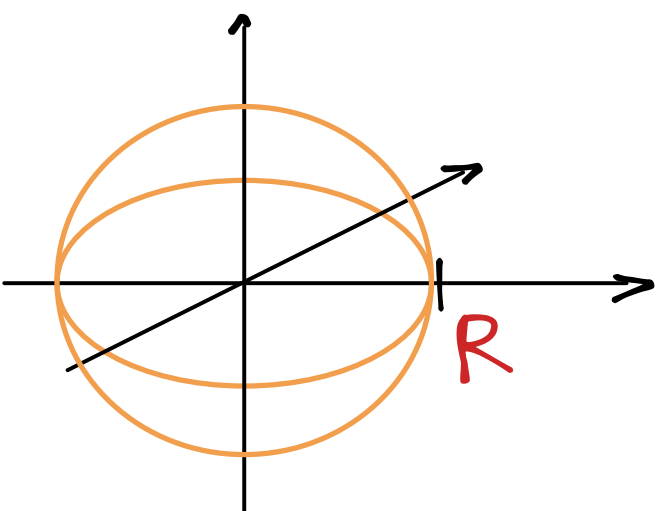
$\Rightarrow \varphi_\alpha(x) = \varphi_\alpha(r)$ for $r = |x|$ and

$$(E_\alpha) \quad \begin{cases} -\Delta \varphi_\alpha = \lambda_\alpha \varphi_\alpha & \text{in } \Omega \\ D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$

reduces to

$$(E_\alpha^r) \quad \begin{cases} -\varphi_\alpha'' - \frac{d-1}{r} \varphi_\alpha' = \lambda_\alpha \varphi_\alpha & \text{on } (0, R); \\ \varphi_\alpha'(R) + \alpha \varphi_\alpha(R) = 0. \end{cases}$$

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Natural conclusion

One might believe that the
1st Robin eigenfunction φ_α is log-concave
for any bounded convex domain & all α .





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One might believe that the
1st Robin eigenfunction φ_α is log-concave
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But this is false!



2. Theorem [Andrews, Clutterbuck, H.'20]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is **NOT** a product of circumsolids.

Then for sufficiently small $\alpha > 0$, the 1st Robin eigenfunction φ_α is not log-concave.



Some Definitions



Some Definitions

- $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, is a convex **polyhedron** if Ω is open, bounded & the intersection of finitely many open half-spaces:

$$\Omega = \bigcap_{i=1}^m \{x \in \mathbb{R}^d \mid x \cdot \gamma_i < b_i\},$$

where $\gamma_1, \dots, \gamma_m$ are unit vectors &
 $b_1, \dots, b_m$ are constants.



Some Definitions

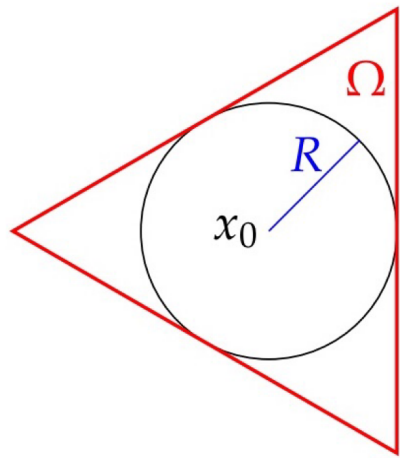
A convex polyhedral domain $\Omega \subseteq \mathbb{R}^d$ is a *circumsolid* if there are $x_0 \in \Omega$ and $R > 0$ s.t. the open ball $B_R(x_0) \subseteq \overline{\Omega}$ touching every face

$$\Sigma_i := \{x \in \overline{\Omega} \mid x \cdot \nu_i = b_i\} \text{ of } \Omega.$$

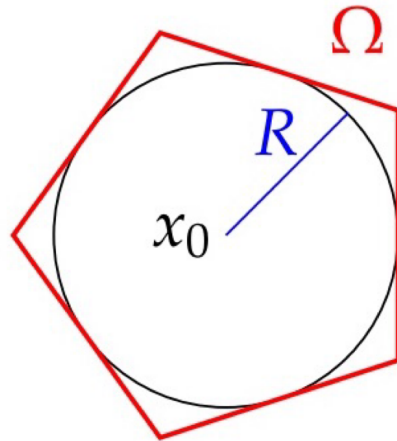
(that is, $\partial B_R(x_0) \cap \Sigma_i$ contains exactly one point)



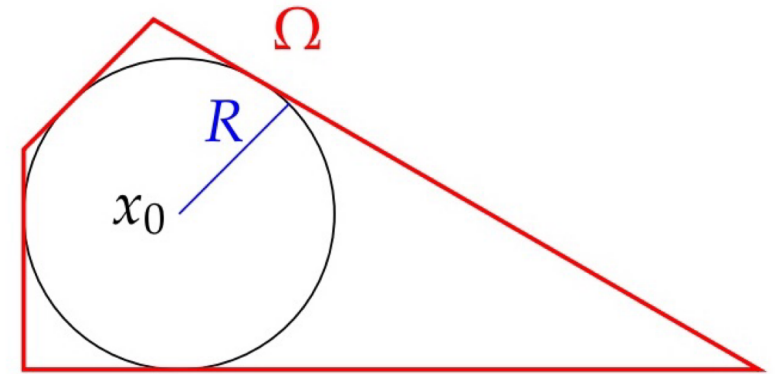
Examples of planar circumsolids:



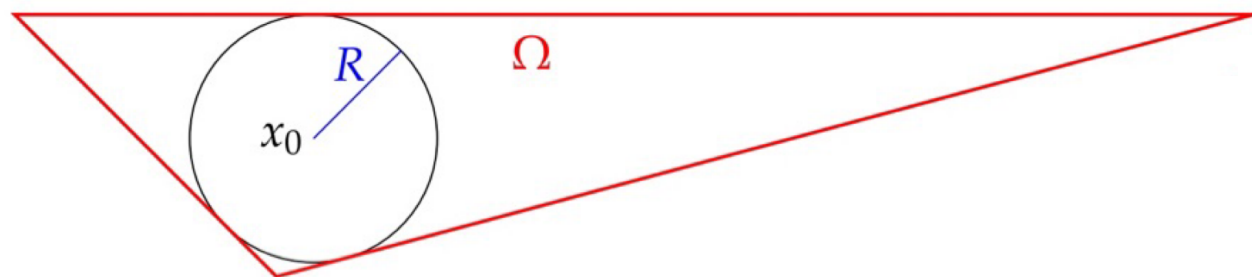
Regular triangle



Regular pentagon



Skew quadrilateral



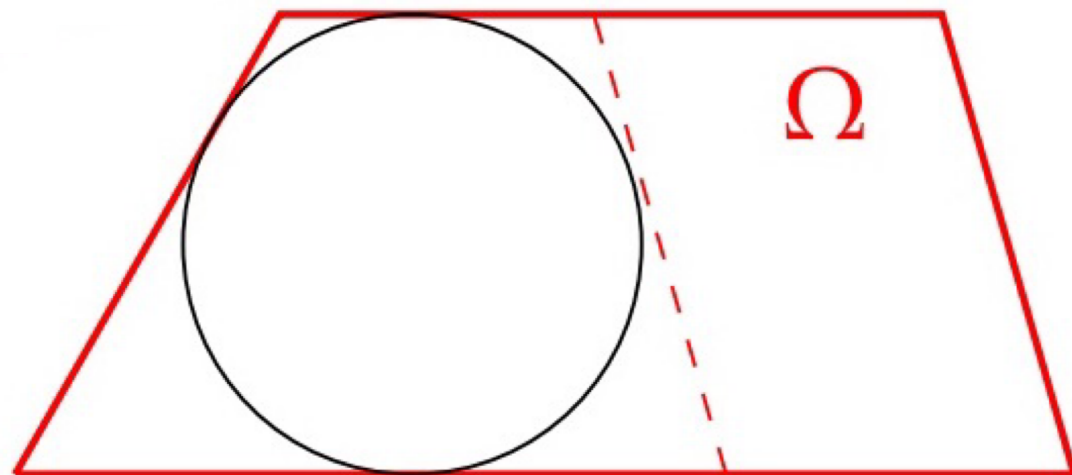
Skew triangle

Every triangle in the plane is a circumsolid.

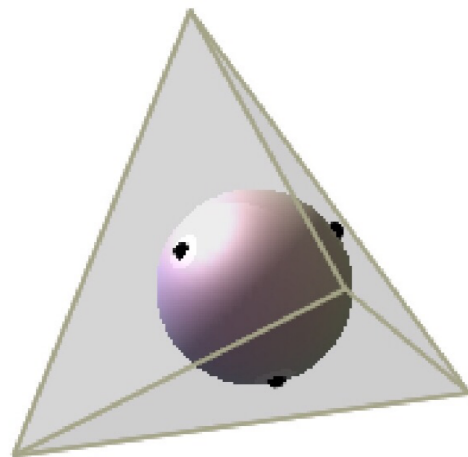


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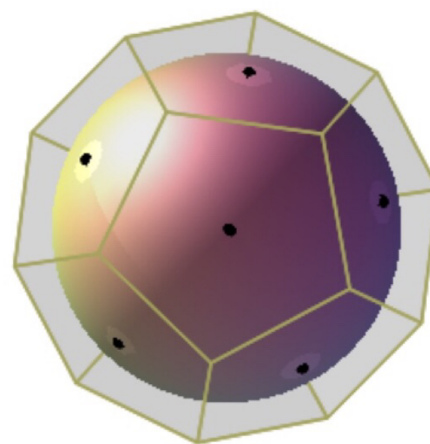
But this is not true for quadrilaterals:



In $\mathbb{R}^3$:



Tetrahedron

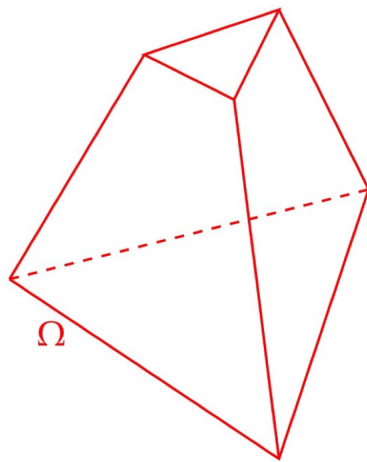


Regular dodecahedron

$$\text{or } \left\{ \sum_{i=0}^d \lambda_i x_i \mid \lambda_i \geq 0, \sum_{i=0}^d \lambda_i = 1 \right\}$$

are all circumsolids.

But



Tetrahedron with a flat top
(cut off the tip!)

is **NOT** a circumsolid

unless the plane of the flat top matches
the inscribed sphere $\partial B_R(x_0)$.

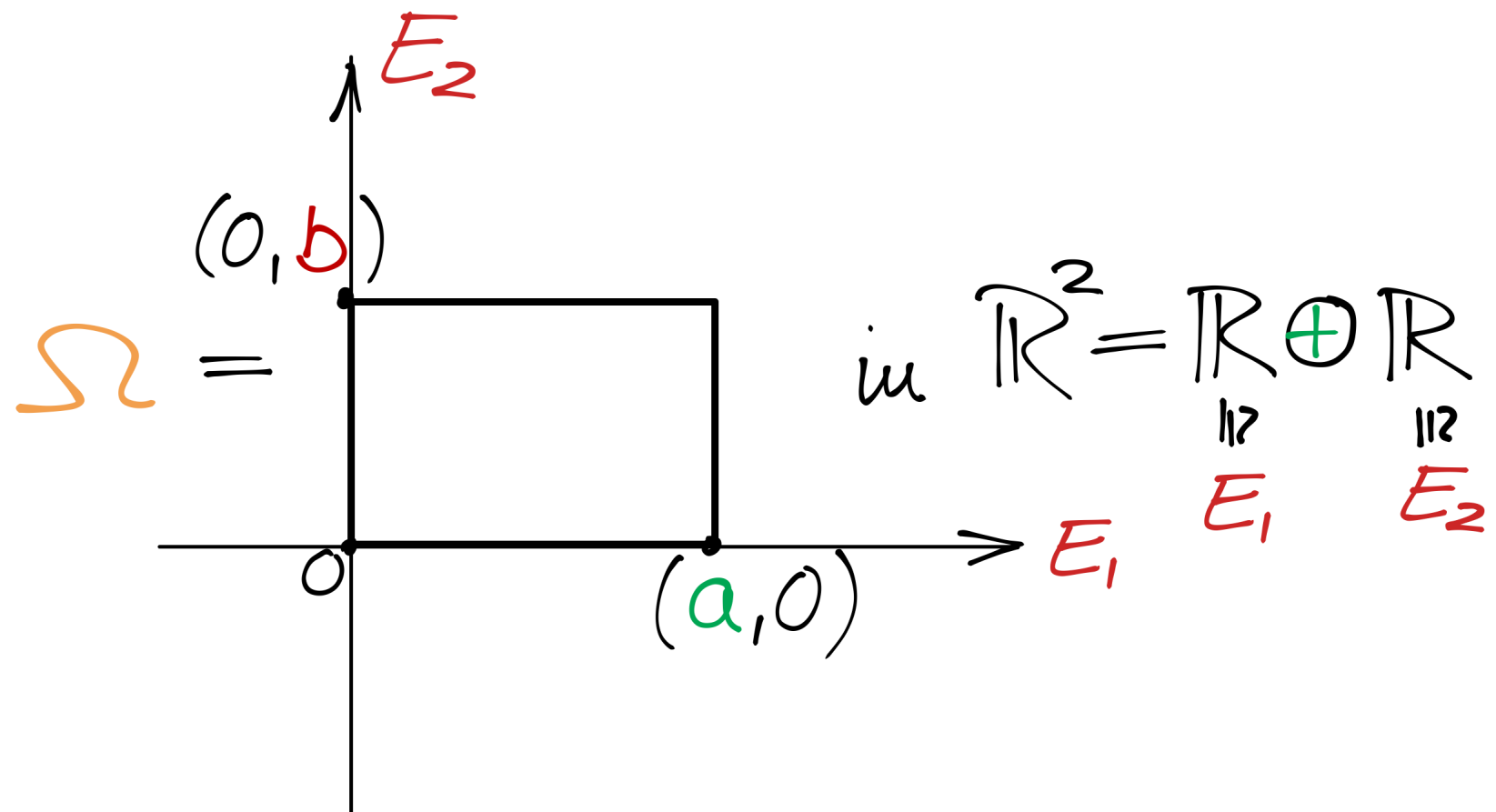
Some Definitions

A convex polyhedral domain $\Omega \subseteq \mathbb{R}^d$ is a *product of circumsolid* if there is a decomposition of $\mathbb{R}^d = \bigoplus_{j=1}^k E_j$ into orthogonal subspaces $E_1, \dots, E_k$ of $\mathbb{R}^d$ and there *circumsolids* $\Omega_1, \dots, \Omega_k$ with $\Omega \subseteq E_i$ such that

$$\Omega = \left\{ x \in \mathbb{R}^d \mid \text{prj}_{E_i}(x) \in \Omega_i \text{ for } i=1, \dots, k \right\}$$

Examples of products of circumsolids

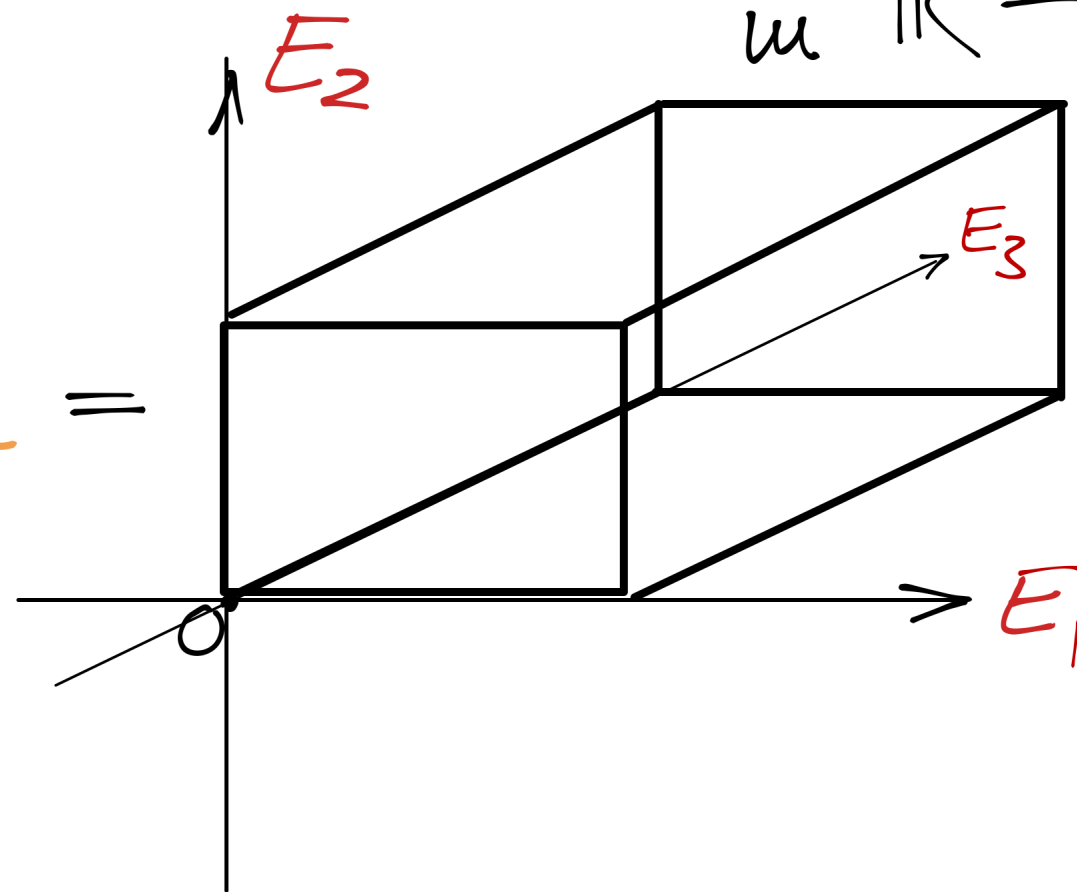
• In $\mathbb{R}^2$: Rectangles



In $\mathbb{R}^3$:

0

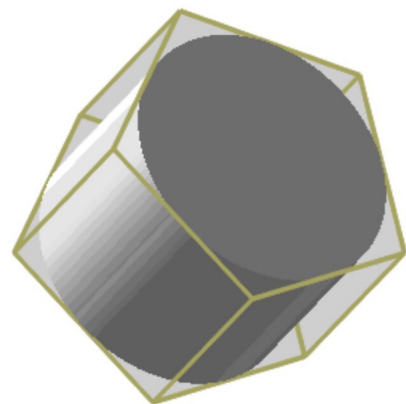
$\Omega =$



$$\text{in } \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$$

$\begin{matrix} \parallel \\ E_1 \end{matrix}$
 $\begin{matrix} \parallel \\ E_2 \end{matrix}$
 $\begin{matrix} \parallel \\ E_3 \end{matrix}$

0



$$\text{in } \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}$$

$\begin{matrix} \parallel \\ E_1 \end{matrix}$
 $\begin{matrix} \parallel \\ E_2 \end{matrix}$

Prism over a regular pentagon.

2. Theorem [Andrews, Clutterbuck, H.'20]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is **NOT** a product of circumsolids.

Then for sufficiently small $\alpha > 0$, the 1st Robin eigenfunction φ_α is not log-concave.



Note: φ_α is not log-concave on Ω

$\iff$ there x & $\hat{x} \in \overline{\Omega}$ & $t_0 \in (0,1)$ s.t.

$$\varphi_\alpha^{t_0}(x) \cdot \varphi_\alpha^{1-t_0}(\hat{x}) > \varphi_\alpha(t_0 x + (1-t_0)\hat{x})$$


Note: φ_α is not *log-concave* on Ω

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If $\Omega_n \rightarrow \Omega$ in Hausdorff & $\Omega_n, \Omega \subseteq \mathcal{D} \subseteq \mathbb{R}^d$
 open & bounded.

$\implies \varphi_{\alpha_n} \cdot 1_{\Omega_n} \rightarrow \varphi_\alpha \cdot 1_\Omega$ in $H^1(\mathcal{D})$ & $\varphi_{\alpha_n} \rightarrow \varphi_\alpha$ in $C^{0,\delta}(\mathcal{B})$ for all $\mathcal{B} \subseteq \bigcap_{n \geq 1} \Omega_n$

1st Corollary [Andrews, Clutterbuck, H. '20]

Let Ω_0 be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is **NOT** a product of circumsolids.

Then for sufficiently small $\alpha > 0$, for any convex domain Ω which is sufficient close to Ω_0 in Hausdorff distance, the 1st Robin eigenfunction φ_α is not log-concave.





Recently, the following result has been shown.



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Theorem [Erasta & Fragala, Camb. J. Math. vol 3. (2021) no.1]

Let $\Omega \subseteq \mathbb{R}^N$ be an open *unif. convex* domain of class $\mathcal{C}^m$ with $[m - \frac{N}{2}] \geq 4$. Then there exist a

threshold $\alpha^* := \alpha^*(D, N, K_1, \dots, K_{N-1}) > 0$

such that for all $\alpha \geq \alpha^*$, the first *Robin*-eigenfunction φ_α of the negative Laplacian $-\Delta$ on Ω is strictly log-concave.





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Problem 3

Can one prove the fundamental gap conjecture for Robin bdy. cond. without using the log-concavity of the first eigenfunction?



$\mathcal{I}_n$ dimension $d=1$:



In dimension $d=1$: Yes !



4. Theorem [Andrews, Clutterbuck, H.'21]

For $\alpha \in \mathbb{R}$, let $\lambda_{\alpha 1}^V$ & $\lambda_{\alpha 2}^V$ be the first & second Robin-eigenvalue of $-\frac{d^2}{dx^2} + V$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

Then for every Symmetric, Single well potential V , the fundamental gap

$$\Pi_{\alpha}^V(V) := \lambda_{\alpha 2}^V - \lambda_{\alpha 1}^V \geq \lambda_{\alpha 2}^0 - \lambda_{\alpha 1}^0 =: \Pi_{\alpha}^V(0)$$

Some remarks



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- (1) Theorem **4** for the Schrödinger operator $-\frac{d^2}{dx^2} + V$ equipped with homogeneous Dirichlet boundary conditions was proved by Ashbaugh & Benguria [**2**].



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- (1) Theorem 4 for the Schrödinger operator $-\frac{d^2}{dx^2} + V$ equipped with homogeneous Dirichlet boundary conditions was proved by Ashbaugh & Benguria [2].
- (2) For the gap $\Gamma_\alpha(0) = \lambda_{\alpha,2}^0 - \lambda_{\alpha,1}^0$, $\lambda_{\alpha,1}^0$ & $\lambda_{\alpha,2}^0$ are the first two roots $\beta > 0$ of the algebraic equation
- $$(\beta^2 - \alpha^2) \tan(\beta\alpha) = 2\alpha\beta.$$

4.* Theorem [Andrews, Clutterbuck, H. '21]

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, let $\lambda_{\alpha 1}^V$ & $\lambda_{\alpha 2}^V$ be the first & second Robin-eigenvalue of $-\{|\alpha_x|^{p-2} \partial_x^2\}_x + V| \cdot |^{p-2}$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

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Proof of Theorem 4^{*}



Proof of Theorem 4.*

- Given a family $\{V_t\}_{t \in \mathbb{R}}$ of potentials (differentiable in t)

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial t} \lambda_{\alpha_i}^{V_t} &= \frac{\partial}{\partial t} \left\{ \int_{\Omega} |u_i|^p + \int_{\Omega} V_t |u_i|^p + \alpha \left(|u_i|_{(-\frac{D}{2})}^p + |u_i|_{(\frac{D}{2})}^p \right) \right\} \\ &= \int_{\Omega} \dot{V}_t |u_i|^p dx \end{aligned}$$

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$u_i :=$ normalized eigenfunction
of $\lambda_{\alpha_i}^{V_t}$

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$$\Rightarrow \frac{\partial}{\partial t} \left(\lambda_{\alpha_2}^{V_t} - \lambda_{\alpha_1}^{V_t} \right) = \int_{\Omega} \dot{V}_t (|u_2|^p - |u_1|^p) dx$$

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& take $V_t := tV \Rightarrow$



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- As usual, $u_1 > 0$ on $\mathcal{J} = (-\frac{D}{2}, \frac{D}{2})$

& u_2 has exactly one root $x_0 \in \mathcal{J}$.

We choose $u_2 > 0$ on $[-\frac{D}{2}, x_0)$.



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We choose $u_2 > 0$ on $[-\frac{D}{2}, x_0)$.

$$\Rightarrow (|u_2|^p - |u_1|^p)(x_0) < 0$$

Proof of Theorem 4.*

◦ Since $\int_{\mathcal{I}} (|u_2|^p - |u_1|^p) dx = 0$

$\Rightarrow |u_2|^p - |u_1|^p$ has at least one root $\xi, \in \mathcal{I}$



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the ratio $\frac{u_2}{u_1}$ is monotonically decreasing on $\mathcal{I}$



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◦ Now, one can show that

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& since

$$(|u_2|^p - |u_1|^p)(\xi) = 0 \quad \text{if} \quad \frac{u_2}{u_1}(\xi) = \pm 1$$

$\Rightarrow |u_2|^p - |u_1|^p$ has at most two roots ξ_1, ξ_2 in $\mathcal{I}$

Proof of Theorem 4.*

Since V is symmetric $\Rightarrow x_0 = 0$ & u_1 is symmetric & u_2 is antisymmetric



Proof of Theorem 4.*

Since V is symmetric $\Rightarrow x_0 = 0$ & u_1 is symmetric & u_2 is antisymmetric
 $\Rightarrow \frac{u_2}{u_1}$ is antisymmetric $\Rightarrow \xi_2 = -\xi_1$ & $\xi_1 < 0 < \xi_2$ & $\xi \in \mathcal{I}$.



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$$\Rightarrow |u_2|^p - |u_1|^p < 0 \quad \text{on} \quad (\xi_1, \xi_2)$$

$$\& \quad |u_2|^p - |u_1|^p > 0 \quad \text{on} \quad \mathcal{I} \setminus [\xi_1, \xi_2].$$

Proof of Theorem 4.*

$$\begin{aligned}
 \Rightarrow \frac{2}{\partial t} \left(\lambda_{\alpha,2}^{V_\epsilon} - \lambda_{\alpha,1}^{V_\epsilon} \right) &= \int_{\mathbb{R}} V \left(|u_2|^p - |u_1|^p \right) dx \\
 &= \int_{\frac{D}{2}}^{\xi_1} \dots dx + \int_{\xi_1}^{\xi_2} \dots dx + \int_{\xi_2}^{\frac{D}{2}} \dots dx \\
 &\geq V(\pm \xi_1) \int_{\mathbb{R}} \left(|u_2|^p - |u_1|^p \right) dx = 0. \quad \square
 \end{aligned}$$

4.* Theorem [Andrews, Clutterbuck, H. '21]

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, let $\lambda_{\alpha 1}^V$ & $\lambda_{\alpha 2}^V$ be the first & second Robin-eigenvalue of $-\{|\alpha_x|^{p-2} \partial_x^2\}_x + V| \cdot |^{p-2}$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

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The general case : V convex



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5. Theorem [Andrews, Clutterbuck, H. '21]

For $\alpha \geq -\frac{1}{2}$, let $\lambda_{\alpha 1}^V$ & $\lambda_{\alpha 2}^V$ be the first & second Robin-eigenvalue of $-\frac{d^2}{dx^2} + V$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

Then for every convex potential V , the fundamental gap

$$\Pi_{\alpha}(V) \geq \Pi_{\alpha}(0).$$

The general case: V convex

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Then for every convex potential V , the fundamental gap

$$\lambda_{\alpha}^V \geq \lambda_{\alpha}^0.$$

Remarks

This result was obtained in the Dirichlet & Neumann by R. Lavine [8].



Idea of the proof of Theorem 5



Idea of the proof of Theorem 5

1st step: Comparison between convex
& linear potentials V



1st step: Comparison between convex
& linear potentials V

6. Theorem [Andrews, Clutterbuck, H. '21]

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, let $\lambda_{\alpha 1}^V$ & $\lambda_{\alpha 2}^V$ be the first & second Robin-eigenvalues of $-\{|\alpha_x| \cdot |\partial_x|^2\}_x + V \cdot |\cdot|^{p-2}$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

Then for every convex potential V , which is not affine, there is $a \in \mathbb{R}$ such that the fundamental gap

$$\lambda_{\alpha}^V \geq \lambda_{\alpha}^a$$



2nd step: Minimization of the fundamental gap
among linear potentials V



2nd step: Minimization of the fundamental gap
among linear potentials V

7. Theorem [Andrews, Clutterbuck, H. '21]

For $\alpha \geq -\frac{1}{2}$, let λ_{α}^V & λ_{α}^V be the first & second Robin-eigenvalues of $-\frac{d^2}{dx^2} + V$ on $\Omega = (-\frac{D}{2}, \frac{D}{2})$.

Among all linear potentials $V_a(x) = ax$, $a \in \mathbb{R}$,

the fundamental gap $\lambda_{\alpha}^V \geq \lambda_{\alpha}^0$ &

"=" holds only when $a = 0$.

Thank You!





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3 Theorem [Andrews, Clutterbuck, H. '20]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$.

- If $d=2$ & Ω is not a product of circumsolids, then the 1st Robin eigenfunction φ_α has non-convex superlevel sets for sufficiently small $\alpha > 0$.
- If $d \geq 3$ & Ω has boundary points with inconsistent normals ν_i , then the same conclusion holds.



Reminders

let Ω be convex.

- A function $\varphi: \Omega \rightarrow \mathbb{R}_+ := (0, \infty)$ is *log-concave* if $v(x) := \log \varphi(x)$ is *concave*

$$\Leftrightarrow \text{for all } x, \hat{x} \in \Omega \text{ \& } \lambda \in (0, 1), \text{ one has that}$$
$$\varphi(\lambda x + (1-\lambda)\hat{x}) \geq \varphi^\lambda(x) \cdot \varphi^{1-\lambda}(\hat{x})$$



Consequence:

φ log-concave $\Rightarrow$
 $\Leftarrow$

for every $c \in \mathbb{R}_+$ the super-level set $E_c := \{x \mid \varphi(x) \geq c\}$ is convex.



Definition

We say that a point $x_0 \in \bar{\Omega}$ has **consistent normals**

if the outward unit normals $\{\nu_i \mid i \in \mathcal{I}(x_0)\}$ to the tangent cone Π_{x_0} lie in a hyperplane disjoint from the origine

$\Leftrightarrow \Pi_{x_0}$ is an (unbounded) circumsolid

$\Leftrightarrow$ there is $\gamma \in \mathbb{R}^d$ solving $\gamma \cdot \nu_i = -1 \quad i \in \mathcal{I}(x_0)$.

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$$\mathcal{I}(x) := \{i \in \{1, \dots, m\} \mid x \cdot \nu_i = b_i\}$$



Definition

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Otherwise, we say that x_0 has **inconsistent normals**.

$$\mathcal{I}(x) := \{i \in \{1, \dots, m\} \mid x \cdot \nu_i = b_i\} \quad \Pi_x^r = \{r(x-y) \in \mathbb{R}^d \mid y \in \Omega, r > 0\}$$

Examples

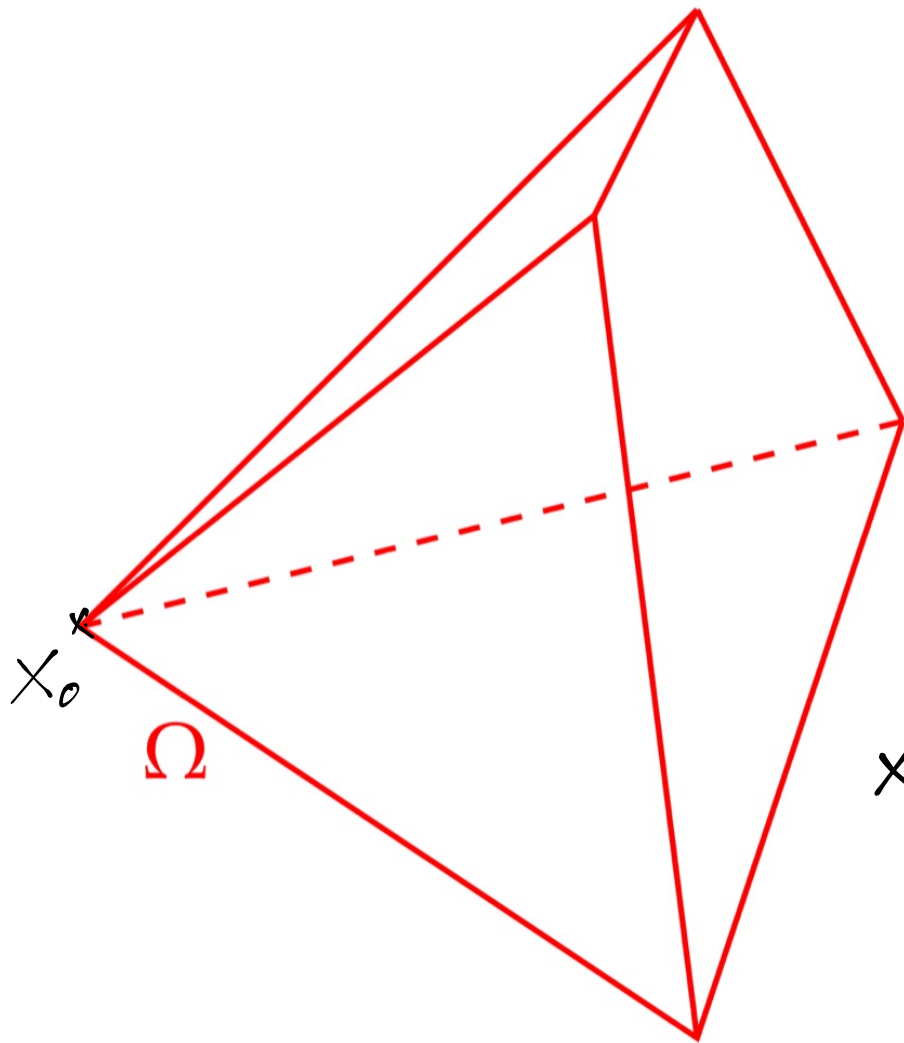
- (1) If Ω is a product of circumsolids, then every boundary point has consistent normals.



Examples

(1) If Ω is a product of circumsolids, then every boundary point has consistent normals.

(2)



x_0 has inconsistent normals.

3 Theorem [Andrews, Clutterbuck, H. '18]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$.

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