# HIGHER-ORDER PAINLEVÉ EQUATIONS IN THE POLYNOMIAL CLASS II. BUREAU SYMBOL P1

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**Abstract.** In this paper, we complete the Painlevé classification of fourth-order differential equations in the polynomial class that was begun in paper I, where the subcase having Bureau symbol $P2$ was treated. This paper treats the more difficult subcase having Bureau symbol $P1$. Some of the calculations involve the use of computer searches to find all cases of integer resonances. Other cases are better handled with the Conte-Fordy-Pickering test. The final list consists of 18 equations denoted F-I, F-II, ..., F-XVIII, built upon 17 distinct reduced equations, one of which (F-II) is not a Painlevé-type equation. The last 12 beginning with F-VII have Bureau symbol $P1$, and the main task of this paper is to prove that there are no others. Equations F-V, F-VI, F-XVII, and F-XVIII define higher-order Painlevé transcendents. Of these, F-VI is new while the other three are group-invariant reductions of the KdV5, the Modified KdV5, and the Modified Sawada-Kotera equations, respectively. Seven of the 18 equations involve hyperelliptic functions of genus 2. We also add to the list another interesting non-Painlevé equation, belonging to a hierarchy of Clarkson and Olver, and give it the number F-XIX. Partial results on the fourth-order classification problem have been obtained previously by Bureau, Exton, and Martynov, the latter author obtaining 13 of the 17 distinct reduced equations with the Painlevé property. Complete solutions are given except in the cases that define the aforementioned higher-order transcendents.

**Key words.** Painlevé analysis, Painlevé transcendents, ordinary differential equations, hyperelliptic functions, nonlinear equations

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## 1. Introduction and Preliminary

In this paper, we carry out the second and final stage of the Painlevé classification of the class of differential equations of the fourth order and first degree in the polynomial class that was begun in paper I [1]. The subcase treated here has Bureau symbol $P1$ (see [2, 3, 4] for the notation) and may be identified by its reduced equations, which take the form,

$$y^{(\mathrm{iv})} = Ayy''' + By'y'' + Cy^2y'' + Dy(y')^2 + Ey^3y' + Fy^5, \tag{1.1}$$

$A$, ..., $F$ being constants, not all zero. (All variables are complex and the prime denotes $d/dx$.) The corresponding full equations take the form,

$$\begin{aligned} y^{(\mathrm{iv})} = {} & A(x)yy''' + B(x)y'y'' + C(x)y^2y'' + D(x)y(y')^2 \\ & + E(x)y^3y' + F(x)y^5 + G(x)y''' + H(x)yy'' \\ & + I(x)(y')^2 + J(x)y^2y' + K(x)y^4 + L(x)y'' \\ & + M(x)yy' + N(x)y^3 + P(x)y' + Q(x)y^2 \\ & + R(x)y + S(x). \end{aligned} \tag{1.2}$$

Our task is to find all equations of the form (1.2) that have the Painlevé property, i.e., whose general solutions admit no movable branch points or branched singular sets. Two equations will be regarded as equivalent if they are related by a gauge transformation of the form,

$$\bar{y} = \lambda(x)y + \mu(x), \qquad \bar{x} = \phi(x). \tag{1.3}$$

Paper I dealt with all other Bureau symbols that can be admitted by a fourth-order differential equation in the polynomial class. The linear equation has symbol $P\infty$. Six equations, denoted F-I, F-II, ..., F-VI, have Bureau symbol $P2$. Among these we intentionally included the Bureau barrier equation, $y^{(\mathrm{iv})} = 3yy'' - 4(y')^2$, which is known to violate some Painlevé tests but which will play an important role in studies of the integrability of differential equations that admit natural barriers. It is the subject of a separate paper [5]. All cases with symbols $P3$ and $P4$ were found to violate a standard test. They admit non-integer resonances. In the next section, we sketch a proof that there are no Painlevé-type equations of the fourth order in the polynomial class outside the classes having the five aforementioned Bureau symbols.

The five Painlevé-type equations having Bureau symbol $P2$, plus the Bureau barrier equation, are given by the following list:

**F-I** : $$y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x + \beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x + \beta)^2; \tag{1.4}$$

**F-II** : $$y^{(\mathrm{iv})} = 3yy'' - 4(y')^2; \tag{1.5}$$

$$\textbf{F-III}: \qquad y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta; \tag{1.6}$$

$$\textbf{F-IV}: \qquad y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta; \tag{1.7}$$

$$\textbf{F-V}: \qquad y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta; \tag{1.8}$$

$$\textbf{F-VI}: \qquad y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta. \tag{1.9}$$

Equation F-I was found by Chazy [6] and is known to arise as a nonclassical reduction of the Boussinesq and KP equations [7]. Equation F-II is the Bureau barrier equation [8, p. 79, (2.9)], which has attracted recent attention from Painlevé analysts [5, 9, 10, 11, 12]. Equations F-III and F-IV were known to Chazy [6, pp. 345,380]. (Actually Chazy found the fifth-order equation, denoted Fif-I in paper I, which generalizes F-III.) The solutions of F-III and F-IV in terms of hyperelliptic functions are given for the first time in paper I. Equation F-V is well known to be a group-invariant reduction of the KdV5 equation. When $\kappa \neq 0$, it is the second member of the Painlevé-I hierarchy. When $\kappa = 0$, its solution in terms of hyperelliptic functions was given by Drach [13].

Equation F-VI was new in paper I and defines a new fourth-order Painlevé transcendent when $\kappa \neq 0$. The corresponding reduced equation can be deduced from earlier results in [14, 15]. Equation F-VI is a group-invariant reduction of two sixth-order partial differential equations discovered by Gordoa and Pickering [16, 17]. It can also be obtained from the following pair of partial differential equations belonging to the Boussinesq hierarchy [18]:

$$\begin{aligned} u_{xxxxx} &= 18uu_{xxx} + 36u_x u_{xx} - 72u^2 u_x - 72vv_x \\ &\qquad + Q(t)\big(u_{xxx} - 12uu_x\big) + 12v_t + R(t), \end{aligned} \tag{1.10}$$

$$v_{xxx} = 6uv_x + 6vu_x + Q(t)v_x - u_t\,. \tag{1.11}$$

The solution of the case $\kappa = 0$ of F-VI in terms of hyperelliptic functions is given in paper I. The hyperelliptic solutions of F-V and F-VI were used by the author [19] to solve the difficult Chazy equations IX and X.

Paper I also contains the solution of the corresponding fifth-order Painlevé classification problem with Bureau symbol $P2$. We found seven equations, denoted Fif-I, Fif-II, Fif-III, Fif-III.b, Fif-IV, Fif-IV.b, and Fif-IV.c, all of which admit first integrals (not necessarily in the polynomial class) and define five distinct fourth-order Painlevé transcendents (including F-V and F-VI). These equations are derivatives or generalizations of derivatives of F-III–F-VI. Particular solutions in terms of classical Painlevé transcendents and hyperelliptic functions are given in paper I. Equations Fif-I and Fif-II are group-invariant reductions of the Kaup-Kupershmidt and Sawada-Kotera equations, respectively. The latter two equations are related to each other by a Bäcklund transformation and have the same modified equation [20]. Equations Fif-III and Fif-III.b are group-invariant reductions of the KdV5 equation. Equations Fif-IV, Fif-IV.b, and Fif-IV.c are new equations. The latter two are derivatives of F-VI, while Fif-IV is a true generalization of F-VI having a first integral of the fourth order and third degree. Additional generalizations of F-VI can be deduced from (1.10) and (1.11).

The problem of classifying fourth-order equations having Bureau symbol $P1$ is somewhat harder because the Diophantine equations that give conditions for integer resonances admit a large number of solutions. The most difficult case is where $F \neq 0$ in the reduced equation (1.1). In that case there are four species of simple poles having three resonance numbers each (not counting the trivial resonance). Let $N_1$, $N_2$, $N_3$, and $N_4$ denote the products of the resonance numbers at each of the four species of poles. Then Chazy [6], Bureau [8], and Martynov [21] obtained the Diophantine equation,

$$\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4} = \frac{1}{24}. \tag{1.12}$$

If cases where one of the $N_i$ takes the value 24 and where pairs $\{N_i, N_j\}$ that satisfy $1/N_i + 1/N_j = 1/24$ are temporarily set aside, then there are exactly 202511 unordered integer solutions of equation (1.12). While this number is large, it is nevertheless finite and every case can be examined by a straightforward computer program. The method that we use is actually much shorter and only uses the values of $N_1$ and $N_2$, which have smaller bounds. Most solutions will be discarded when it found that the individual resonance numbers are not all integers. (We used the computer algebra package REDUCE to do calculations of this type.)

However, the case where one of the $N_i$ equals 24 presents special difficulties by this method. There are now infinitely many solutions to examine, and they cannot be conveniently indexed by an integer parameter. For a while we called this case the "intractable case," not because it gave us any problems, but because it could not be analysed in full by the methods employed by earlier authors. In Section 5 below, we show that this case can be handled thoroughly and quite efficiently by an application of the perturbative Painlevé test (the "CFP test") of Conte, Fordy, and Pickering [9, 10]. The CFP test can also be used to shorten the analysis of the cases where one of the $N_i$ differs from 24 by a small integer.

We present here the results of the Painlevé classification carried out in this paper for the case where the Bureau symbol is $P1$. We obtain 11 reduced equations and 12 full equations, F-VII–F-XVIII, built upon these reduced equations (F-X and F-XI having the same reduced equation). Up to the family of gauge transformations (1.3), these 12 equations are unique. Together with F-I, F-III–F-VI, and the fourth-order linear equation, they give the complete Painlevé classification of fourth-order differential equations in the polynomial class. Canonical representatives of the equations F-VII–F-XVIII are given by

$$\begin{aligned}
\textbf{F-VII}: \qquad y^{(\mathrm{iv})} = & -\frac{3f'}{f}\,y''' - \frac{ff''+(f')^2}{f^2}\,y'' \\
& - \frac{4}{f^2}\Big\{3c_1x^2(xy'-y) + c_2x(3xy'-2y) \\
& + c_3(3xy'-y) + 3c_4y' + c_5x^2 + c_6x + c_7\Big\}y'' \\
& - \frac{2}{f^2}\Big\{3c_1(xy'-y)^2 + 2c_2y'(xy'-y) + c_3(y')^2 \\
& + 2c_5(xy'-y) + c_6y' + c_8\Big\},
\end{aligned} \tag{1.13}$$

where $f(x) := c_1x^3 + c_2x^2 + c_3x + c_4$;

$$\begin{aligned}
\textbf{F-VIII}: \qquad y^{(\mathrm{iv})} = & -2yy''' - 6y'y'' + A(x)\big\{y''' + 2yy'' + 2(y')^2\big\} \\
& + B(x)\big(y'' + 2yy'\big) + C(x)\big(y' + y^2\big) + D(x);
\end{aligned} \tag{1.14}$$

$$\begin{aligned}
\textbf{F-IX}: \qquad y^{(\mathrm{iv})} = & -3yy''' - 9y'y'' - 3y^2y'' - 6y(y')^2 \\
& + A(x)\big\{y''' + 3yy'' + 3(y')^2 + 3y^2y'\big\} \\
& + B(x)\big\{y'' + 3yy' + y^3\big\} + C(x);
\end{aligned} \tag{1.15}$$

$$\begin{aligned}
\textbf{F-X}: \qquad y^{(\mathrm{iv})} = & \frac{d}{dx}R(x,y,y',y'') + \frac{1}{x}\Big\{y''' - R(x,y,y',y'')\Big\} \\
& + \lambda x\big\{y' + y^2 - g(x)\big\} - \tfrac{1}{96}(N^2-1)\lambda^2x^3;
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
\textbf{F-XI}: \qquad y^{(\mathrm{iv})} = & \frac{d}{dx}R(x,y,y',y'') \\
& + \lambda\big\{y' + y^2 - g(x)\big\} - \tfrac{1}{24}(N^2-1)\lambda^2x,
\end{aligned} \tag{1.17}$$

where

$$\begin{aligned}
R(x,y,y',y'') := & -2yy'' - 2(y')^2 + g''(x) + \tfrac{1}{4}(N^2-1)(\alpha x + \beta) \\
& + [24/(N^2-1)]\big\{y' + y^2 - g(x)\big\}^2;
\end{aligned}$$

$$
\textbf{F-XII}: \quad y^{(\mathrm{iv})} = -4yy''' - 10y'y'' - 6y^2y'' - 12y(y')^2 - 4y^3y' + A(x)\{y'' + 2yy'\} + \{B(x) + A'(x)\}y' + A'(x)y^2 + B'(x)y + C(x); \tag{1.18}
$$

$$
\textbf{F-XIII}: \quad y^{(\mathrm{iv})} = -3yy''' - 10y'y'' - 4y^2y'' - 8y(y')^2 - 2y^3y' + 2A(x)\{y'' + yy'\} + A'(x)\{3y' + y^2\} + A''(x)y + C(x); \tag{1.19}
$$

$$
\textbf{F-XIV}: \quad y^{(\mathrm{iv})} = \frac{d}{dx}\Big\{12(y')^2 + 72y^2y' + 54y^4\Big\}; \tag{1.20}
$$

$$
\textbf{F-XV}: \quad y^{(\mathrm{iv})} = -2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y' + 3R(x)(y'' + yy') + R'(x)(4y' + y^2) + R'''(x) - 2R(x)R'(x); \tag{1.21}
$$

$$
\textbf{F-XVI}: \quad y^{(\mathrm{iv})} = -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5 + A(x)\big(y''' + 4yy'' + 3(y')^2 + 6y^2y' + y^4\big) + B(x)(y'' + 3yy' + y^3) + C(x)(y' + y^2) + D(x)y + E(x); \tag{1.22}
$$

$$
\textbf{F-XVII}: \quad y^{(\mathrm{iv})} = 10y^2y'' + 10y(y')^2 - 6y^5 - 10\delta(y'' - 2y^3) + (\lambda x + \mu)y + \sigma; \tag{1.23}
$$

$$
\textbf{F-XVIII}: \quad y^{(\mathrm{iv})} = -5y'y'' + 5y^2y'' + 5y(y')^2 - y^5 + (\lambda x + \alpha)y + \gamma. \tag{1.24}
$$

To this list, we would like to add a non-Painlevé equation of similar shape that we anticipate will be of considerable interest to researchers in the analytic theory of differential equations. It is the simplest example beyond the Chazy-XII equation (i.e., Chazy's natural barrier equation) that can be derived and solved by the theory of Clarkson and Olver [22]. We call it the Clarkson-Olver equation and give it the number F-XIX. The equation is

$$
\textbf{F-XIX}: \quad \left(\frac{d}{dx} - \tfrac{4}{3}y\right)\left\{y''' - 2yy'' + 3(y')^2 - \alpha\big(6y' - y^2\big)^2\right\} = 0, \tag{1.25}
$$

where $\alpha$ is an arbitrary complex constant. Setting the quantity inside the braces to zero gives the Chazy-XII equation (Chazy-III when $\alpha = 0$). The latter equation has a single-valued general solution when $\alpha = 4/(36 - N^2)$, where $N$ is a positive integer distinct from 1 or 6, and when $\alpha = 0$. In general, the Clarkson-Olver equation has a very complicated singularity structure, but is integrable parametrically in terms of Lamé functions. We regard this equation as integrable because its general solution can be constructed. However, in general, it fails Kruskal's poly-Painlevé test for integrability [23], and so some authors would prefer a different adjective such as "solvable." The general solution of F-XIX and some remarks about the Clarkson-Olver hierarchy to which it belongs are included in Section 7. One result of particular interest is the following generalization of Chazy-XII and F-XIX:

$$\left(\frac{d}{dx} - \tfrac{5}{3}y\right)\left(\frac{d}{dx} - \tfrac{4}{3}y\right)\left\{y''' - 2yy'' + 3(y')^2 - \alpha\big(6y' - y^2\big)^2\right\} = 0. \tag{1.26}$$

The solution of this equation involves the Painlevé-I transcendent.

Another equation closely related to the Chazy-XII equation is

$$\left(\frac{d}{dx} + y\right)\left\{y''' - 2yy'' + 3(y')^2 + \tfrac{1}{16}\big(6y' - y^2\big)^2\right\} = 0. \tag{1.27}$$

This equation is new and is a direct byproduct of the Painlevé analysis in Section 4 below. It has integer resonance numbers but fails a test for the absence of logarithms at one of those resonances. It admits a three-parameter single-valued solution with a natural barrier in the complex plane that satisfies the Chazy-XII equation with $N = 10$.

The classical work of Chazy [6] and Bureau [8, 24] on the fourth-order classification problem with Bureau symbol $P1$ was confined to the cases having $E = F = 0$ in the reduced equation (1.1). Chazy found F-VII and Bureau found F-VII–F-IX and studied them in detail. Their work was incomplete because a number of cases could not be analysed with their methods. These included the derivative of the Chazy-III equation, $y''' = 2yy'' - 3(y')^2$, and the equations of the form (3.2) below. The former case was handled by Pickering and coworkers [9, 10] and the latter equations were discussed in part by Martynov [21].

The cases where $E$ and $F$ are not both zero were attempted by Exton [25] and Martynov [21]. The constant $E$ is nonzero while $F = 0$ in the cases F-X–F-XV and the constant $F$ is nonzero in the cases F-XVI–F-XVIII. Exton's methods were flawed and he rejected many good Painlevé-type equations, including F-IV and the third-order equations Chazy-IX and Chazy-X [19]. He obtained F-XI (with incorrect constraints on the coefficients), F-XII, F-XIII, and F-XVI. Martynov worked with the Diophantine equations giving conditions for integer resonances, including (1.12) above, but did not show details. His list is the most complete so far. However, he missed four reduced equations, namely, F-III, F-VI, F-XV, and F-XVIII.

## 2. Preliminary steps in the Painlevé classification

We give here an outline of the proof that the Painlevé classification of fourth-order equations in the polynomial class can be restricted to cases having a well defined Bureau symbol of $Pk$ type. Some of the steps are similar to those in [2, 3, 4, 26, 27] for second-order equations. Previous authors [6, 8, 21, 25, 28] have skipped over these arguments, but none of them have suggested that the arguments are all straightforward enough to be left to the interested reader. These authors were aware of certain problematic cases whose Painlevé analysis is not obvious. We present below a new method for dealing with such cases.

The starting point is the class of equations,

$$y^{(\mathrm{iv})} = F(x, y, y', y'', y'''), \tag{2.1}$$

where $F$ is a polynomial in $y$ and its first three derivatives and is locally analytic in $x$. The first few steps can be carried out under the more general hypothesis that $F$ is rational in $y$ and its derivatives. We shall examine the rational case up to the point where we arrive at the fundamental reduced equation (2.17) below.

If $F$ is rational and has a factor $\big(y''' - g(x, y, y', y'')\big)^k$ in its denominator, where $g$ is rational or algebraic in $y$ and its first two derivatives and $k \geq 1$, then the Painlevé $\alpha$-test,

$$\begin{aligned} x &= x_0 + \alpha X, \\ y &= y_0 + \alpha a X + \tfrac{1}{2}\alpha^2 b X^2 + \tfrac{1}{6}\alpha^3 g(x_0, y_0, a, b) X^3 + \alpha^{(3k+4)/(k+1)} Y, \end{aligned} \tag{2.2}$$

gives, in the limit $\alpha \to 0$, the reduced equation,

$$Y^{(\mathrm{iv})} = D_0 \big(Y'''\big)^{-k}, \tag{2.3}$$

where the prime denotes $d/dX$ and the constant $D_0$ will be nonzero for generic choices of the arbitrary constants $x_0$, $y_0$, $a$, and $b$. The reduced equation is easily solved and its general solution admits a movable algebraic branch point of degree $k+1$. This implies that the function $F$ in (2.1) can have no factor of the postulated type in its denominator and hence cannot have $y'''$ appearing in its denominator. Thus $F$ is a polynomial in its last argument $y'''$.

The next step is to show that $F$, regarded as a polynomial in $y'''$, can have degree at most 2. Suppose, on the contrary, that it has degree $k \geq 3$. Then the Painlevé $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = y_0 + \alpha a X + \tfrac{1}{2}\alpha^2 b X^2 + \alpha^{(3k-4)/(k-1)} Y, \tag{2.4}$$

gives the reduced equation,

$$Y^{(\mathrm{iv})} = D_0 \big(Y'''\big)^k, \tag{2.5}$$

whose general solution admits a movable algebraic branch point of degree $k-1$. The equation under investigation is now restricted to the form,

$$y^{(\mathrm{iv})} = A(x,y,y',y'')(y''')^2 + B(x,y,y',y'')y''' + C(x,y,y',y''), \tag{2.6}$$

where the coefficients are rational in $y$, $y'$, and $y''$ and locally analytic in $x$.

The coefficient $A$ can be singled out for special attention by the $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = y_0 + \alpha a X + \alpha^2 Y. \tag{2.7}$$

The resulting reduced equation is

$$Y^{(\mathrm{iv})} = A(x_0, y_0, a, Y'')\big(Y'''\big)^2 =: A(Y'')\big(Y'''\big)^2, \tag{2.8}$$

where, for generic choices of $x_0$, $y_0$, and $a$, $A(Y'')$ is a nonzero rational function of its argument. Now we may let $Z = Y''$. The class of differential equations,

$$Z'' = A(Z)(Z')^2, \tag{2.9}$$

arises in a similar fashion in the corresponding Painlevé classification problems of all differential orders, including partial differential equations in any number of independent variables. Its Painlevé analysis is well known and is given in [2, 3, 4, 26, 27]. The function $A(Z)$ cannot have poles of order higher than the first and it must obey $A(Z) = \mathrm{O}(1/Z)$ as $Z \to \infty$. At simple poles of $A(Z)$ the residues must be either 1 or $(m-1)/m$ for some nonzero integer $m$ and the sum of the residues must also take one of these forms. These conditions restrict $A(Z)$ to a list of 21 particular rational functions (see [3, p. 19]).

In second-order classification problems, the aforementioned 21 forms for $A(Z)$ can be restricted to 8 canonical forms by exploiting suitable Möbius transformations in $Z$. This freedom is not available in higher-order problems because $Z$ stands for a derivative of $Y$. Now, in 8 of the 21 forms for $A(Z)$, the sum of the residues is 2, in which case an obvious $\alpha$-test reduces the equation to $Z'' = (2/Z)(Z')^2$, whose solution is $Z = K/(X - X_0)$. In higher-order problems, these 8 cases are not permitted because $Y$ has a movable logarithmic branch point. In particular, the third-order classification problem must examine 13 possible forms for $A(Z)$, which would correspond to a coefficient of the form $A(x,y,y')$ in the full equation. In the fourth-order problem, forms of $A(Z)$ in which the sum of the residues is $3/2$ must also be eliminated because $Y''$ has a movable double pole in these cases and $Y$ has a movable logarithmic branch point. More generally, forms of $A(Z)$ in which the sum of the residues is $(p{+}1)/p$, $p \geq 1$, are not permitted in Painlevé classification problems where the differential order is $p{+}2$ or higher.

The foregoing arguments leave the following 9 permitted forms for the function $A(x, y, y', y'')$ in (2.6):

$$0,$$

$$\frac{1}{y'' - g(x, y, y')},$$

$$\frac{m-1}{m\{y'' - g(x, y, y')\}} \quad (m \geq 2),$$

$$\frac{m+1}{m\{y'' - g(x, y, y')\}} \quad (m \geq 3),$$

$$\frac{2y'' + g(x, y, y')}{2\{(y'')^2 + g(x, y, y')y'' + h(x, y, y')\}},$$

$$\frac{2\{2y'' + g(x, y, y')\}}{3\{(y'')^2 + g(x, y, y')y'' + h(x, y, y')\}},$$

$$\frac{3}{4\{y'' - g(x, y, y')\}} + \frac{1}{2\{y'' - h(x, y, y')\}},$$

$$\frac{2}{3\{y'' - g(x, y, y')\}} + \frac{1}{2\{y'' - h(x, y, y')\}},$$

$$\frac{5}{6\{y'' - g(x, y, y')\}} + \frac{1}{2\{y'' - h(x, y, y')\}},$$

where $g(x, y, y')$ and $h(x, y, y')$ are rational functions of $y$ and $y'$.

The next step is to show that all the poles of $B$ and $C$, regarded as rational functions of $y''$, are simple and included among the poles of $A$. Suppose, on the contrary, that at least one of $B$ and $C$ admits a higher-order pole at $y'' = g(x, y, y')$, where $g$ is rational or algebraic in $y$ and $y'$. Thus $A$, $B$, and $C$ admit ascending power series in the variable $y'' - g(x, y, y')$ with leading terms,

$$
\begin{aligned}
A &= \frac{a_0}{y'' - g(x,y,y')} + \dots, \\
B &= \frac{b(x,y,y')}{\{y'' - g(x,y,y')\}^p} + \dots, \\
C &= \frac{c(x,y,y')}{\{y'' - g(x,y,y')\}^q} + \dots,
\end{aligned}
$$

where $a_0$ is a constant, which is allowed to be zero, and either $p \geq 2$ or $q \geq 2$ or both. Without loss of generality, we may set $q = 2p - 1$ on the understanding that $p$ can be an integer or half-odd-integer $\geq 3/2$ and one, but not both, of $b$ and $c$ can be the zero function. The Painlevé $\alpha$-test,

$$
\begin{aligned}
x &= x_0 + \alpha X, \\
y &= y_0 + \alpha y_1 X + \tfrac{1}{2}\alpha^2 g_0 X^2 + \alpha^{2+1/p} Y,
\end{aligned}
\tag{2.10}
$$

where $g_0 := g(x_0, y_0, y_1)$, yields the reduced equation,

$$
Y^{(\mathrm{iv})} = \frac{a_0}{Y''}\left(Y'''\right)^2 + \frac{b_0}{(Y'')^p}\,Y''' + \frac{c_0}{(Y'')^{2p-1}}\,, \tag{2.11}
$$

where $b_0 = b(x_0, y_0, y_1)$ and $c_0 = c(x_0, y_0, y_1)$, these constants being not both zero. On setting $Z = Y''$, we arrive at a second-order equation which is known to admit movable algebraic branch points [2, 3, 4, 26, 27]. Hence all poles of $B$ and $C$ in $y''$ must be simple.

In the case where the poles of $B$ and $C$ are all simple, choose the $\alpha$-test,

$$
\begin{aligned}
x &= x_0 + \alpha X, \\
y &= y_0 + \alpha y_1 X + \tfrac{1}{2}\alpha^2 g_0 X^2 + \alpha^3\{Y + \tfrac{1}{6}h_0 X^3\},
\end{aligned}
\tag{2.12}
$$

where $h_0 = \partial g/\partial x + y_1 \partial g/\partial y + g\partial g/\partial y'$ evaluated at $(x, y, y') = (x_0, y_0, y_1)$. The reduced equation takes the form,

$$
Y^{(\mathrm{iv})} = \frac{a_0}{Y''}\left(Y''' + h_0\right)^2 + \frac{b_0}{Y''}\left(Y''' + h_0\right) + \frac{c_0}{Y''}\,, \tag{2.13}
$$

where $b_0$ and $c_0$ are constants, not both zero. After setting $Z = Y''$, we arrive at an equation which is known to admit logarithmic branch points whenever $a_0 = 0$. It follows that all the poles of $B$ and $C$ in $y''$ must be included among the poles of $A$.

There are also degree restrictions on $B$ and $C$, regarded as polynomials or rational functions of $y''$. We already know that $A \sim a_0/y''$ as $y'' \to \infty$, where $a_0$ is either 1 or $(m-1)/m$ for some integer $m \neq 0, -1, -2$. Suppose $B$ and $C$ admit descending power series in the variable $y''$ with leading terms,

$$\begin{aligned} B &= b(x, y, y')\big(y''\big)^p + \dots, \\ C &= c(x, y, y')\big(y''\big)^q + \dots . \end{aligned}$$

We want to show that $p \leq 1$ and $q \leq 3$. Suppose that one or both of these inequalities does not hold. Without loss of generality, we may set $q = 2p + 1$ on the understanding that $p$ can be an integer or half-odd-integer $\geq 3/2$ and one, but not both, of $b$ and $c$ can be the zero function. The Painlevé $\alpha$-test,

$$\begin{aligned} x &= x_0 + \alpha X, \\ y &= y_0 + \alpha y_1 X + \alpha^{2-1/p} Y, \end{aligned} \tag{2.14}$$

yields the reduced equation,

$$Y^{(\mathrm{iv})} = \frac{a_0}{Y''}\big(Y'''\big)^2 + b_0\big(Y''\big)^p Y''' + c_0\big(Y''\big)^{2p+1}, \tag{2.15}$$

where $b_0$ and $c_0$ are constants, not both zero. Again, after setting $Z = Y''$, we arrive at an equation which is known to admit movable algebraic branch points. Hence the degrees of $B$ and $C$ satisfy $p \leq 1$ and $q \leq 3$.

So far, we have restricted rational fourth-order equations having the Painlevé property to the form (2.6) with $A(x, y, y', y'')$ taking one of the nine forms listed above, with $B(x, y, y', y'')$ and $C(x, y, y', y'')$ having at most simple poles in $y''$ which must be included among the poles of $A(x, y, y', y'')$, and with the degree restrictions $B = \mathrm{O}(y'')$ and $C = \mathrm{O}\big((y'')^3\big)$ as $y'' \to \infty$. Further constraints on the behaviors of $B$ and $C$, regarded as rational functions of $y''$, can be deduced by methods analogous to the analysis of the Painlevé-Gambier class of equations in [2, 27, 29]. The manner in which $x$, $y$, and $y'$ can enter the coefficients is a difficult open problem in Painlevé analysis and very little has been done on this problem or on the corresponding third-order problem.

The admissible behaviors of $B(x, y, y', y'')$ and $C(x, y, y', y'')$ when $y''$ is large can be deduced by applying the $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = y_0 + \alpha Y. \tag{2.16}$$

We arrive at what we call the "fundamental reduced equation,"

$$Y^{(\mathrm{iv})} = \frac{m-1}{mY''}\big(Y'''\big)^2 + P(Y')Y''Y''' + Q(Y')\big(Y''\big)^3, \tag{2.17}$$

where $m$ is either infinite or an integer distinct from 0, $-1$, or $-2$. This equation can always be solved in terms of a second-order linear differential equation, a functional inversion, and quadratures. The coefficients $P(Y')$ and $Q(Y')$ are rational functions to be determined. If we write $Z = Y'$, the class of equations (2.17) becomes a class that has been studied by Painlevé [26, 30], Chazy [6], Garnier [28], and Carton-Lebrun [31, 32, 33]. Among the constraints to be satisfied by $P(Z)$ and $Q(Z)$ are ($i$) the poles of $P$ are simple and the poles of $Q$ are simple or double, ($ii$) the poles of $Q$ are included among the poles of $P$ except when $m = -2$, in which case $P = 0$ and $Q$ has any number of double poles, ($iii$) $P(Z) = \mathrm{O}(1/Z)$ and $Q(Z) = \mathrm{O}(1/Z^2)$ as $Z \to \infty$. Additional constraints arise from conditions for the absence of logarithms at positive-integer resonances, if any. There is a finite list of admissible equations, some containing arbitrary integer parameters, where two equations are counted as equivalent if one can be transformed into the other by a Möbius transformation in $Z$. Our own (unpublished) counting rules give a list of 96 equations. (Of these, 10 equations have solutions admitting movable essential singularities or natural barriers.) All of these equations are explicitly solvable in terms of elliptic or simpler functions except one master equation solvable in terms of Fuchsian and Kleinian functions. This master equation has $m = -2$ and so is not admissible in the corresponding fourth-order problem.

When $Z = Y'$, the Möbius freedom in $Z$ is not a gauge transformation acting on the fourth-order equations for $Y$. Instead, one must look among the images of $Z$ under Möbius transformations $Z \to (aZ + b)/(cZ + d)$ to see which ones have $Y = \int Z(X)\, dX$ single-valued. Only a small number, if any, of discrete Möbius transformations are admissible because a generic Möbius transformation creates simple poles in $Z(X)$ and hence logarithmic branch points in $Y(X)$. For the same reason, we also have to exclude cases where $Z(X)$ has higher-order poles or isolated essential singularities with nonzero residue.

In a preliminary search, we found that 38 of the 96 cases admitted no single-valued integrals, 55 admitted one distinct single-valued integral (including cases where more than one Möbius transformation gave a single-valued integral, but the integrals so obtained were similar to each other), and 3 admitted two distinct single-valued integrals. In the cases involving arbitrary integer parameters, the condition for a single-valued integral either eliminated those cases or placed restrictions on the integer parameters, such as requiring them to belong to certain arithmetic progressions. It follows that, according to our counting rules, the rational fourth-order Painlevé classification problem must take into account 61 fundamental reduced equations of the form (2.17). (Of these, two have solutions admitting movable essential singularities.)

Since this paper is concerned with the polynomial class, the only case of relevance to us is $A(x, y, y', y'') = 0$. Among the 61 aforementioned fundamental reduced equations, 37 have $m = 1$ and correspond to large classes of full equations having $A(x, y, y', y'') = 0$. Only one of these classes has $B$ and $C$ being polynomials in $y'$ as well as $y''$. Since $P(Y')$ and $Q(Y')$ in (2.17) always decay to zero as $Y' \to \infty$, these coefficients cannot be nonzero polynomials. The only possibility is for these coefficients to be identically zero, in which case the fundamental reduced equation reduces to $Y^{(\mathrm{iv})} = 0$. It follows that $B$ and $C$, if they are polynomials in both $y'$ and $y''$, must have maximum degrees 0 and 2, respectively, in the variable $y''$. This is true regardless of whether $B$ and $C$ are polynomials or rational functions of the variable $y$. From now on, we restrict attention to the case where $B$ and $C$ are polynomials in $y$, $y'$, and $y''$.

So far, we have shown that fourth-order Painlevé-type equations in the polynomial class must be restricted to the form,

$$y^{(\mathrm{iv})} \;=\; B(x, y, y')y''' + D(x, y, y')(y'')^2 + E(x, y, y')y'' + F(x, y, y'), \tag{2.18}$$

where $B$, $D$, $E$, and $F$ are polynomials in $y$ and $y'$. We will see below that $D$ must also be zero.

Next, suppose that $B$, $D$, $E$, and $F$ have degrees $p$, $q$, $r$, and $s$, respectively, when regarded as polynomials in the variable $y'$. If at least one of the inequalities,

$$p \geq 2, \qquad q \geq 1, \qquad r \geq 3, \qquad s \geq 5, \tag{2.19}$$

holds, then the $\alpha$-test,

$$x \;=\; x_0 + \alpha X, \qquad y \;=\; y_0 + \alpha^t Y, \tag{2.20}$$

$$t \;=\; \max\left\{\frac{p-1}{p}, \frac{q}{q+1}, \frac{r-2}{r}, \frac{s-4}{s-1}\right\},$$

will yield a reduced equation of one of the following three forms:

$$\begin{aligned} Y^{(\mathrm{iv})} \;=\;& B_0\big(Y'\big)^p Y''' + D_0\big(Y'\big)^{p-1}\big(Y''\big)^2 \\ &+ E_0\big(Y'\big)^{2p} Y'' + F_0\big(Y'\big)^{3p+1} \qquad (p \geq 2), \end{aligned} \tag{2.21}$$

$$Y^{(\mathrm{iv})} \;=\; E_0\big(Y'\big)^r Y'' \qquad (r \geq 3), \tag{2.22}$$

$$Y^{(\mathrm{iv})} \;=\; F_0\big(Y'\big)^s \qquad (s \geq 5). \tag{2.23}$$

In (2.21), any of the coefficients can be zero except that not all can be zero simultaneously. In (2.22) and (2.23), the single coefficient in each case is nonzero.

Equations (2.21)–(2.23) are third-order equations for the variable $Z := Y'$. In all cases except one, the differential equation admits an exact solution of the form,

$$Y \;=\; a_0\big(X - X_0\big)^\rho, \tag{2.24}$$

where $X_0$ is a constant of integration, $a_0 \neq 0$, and $0 < \rho < 1$. This particular integral can be embedded in the general integral in various ways, for example, by an ascending series of non-integer powers of $X - X_0$, possibly supplemented by logarithms, if all the resonance numbers are positive or have positive real part, or by a CFP perturbation expansion [9, 10]. It is not necessary to carry out these constructions because (2.24) is guaranteed to be a member of the general integral because no rational or polynomial members of the class (2.1) possess singular integrals. Hence all equations of the form (2.21)–(2.23), with one exception, violate the Painlevé property for the simple reason that $\rho$ is not an integer.

The exceptional case not possessing an exact solution of the form (2.24) is equation (2.21) with

$$D_0 \;=\; -(p+1)B_0\,, \qquad E_0 \;=\; 0, \qquad F_0 \;=\; 0. \tag{2.25}$$

When we replace $Y'$ by $Z$ and $p$ by $n$, we obtain Chazy's problematic equation [6, 34, 35],

$$Z''' \;=\; B_0\big\{Z^n Z'' - (n+1)Z^{n-1}(Z')^2\big\}. \tag{2.26}$$

A solution involving two integration constants can be written in the form of a formal double series of descending powers of $\log(X - X_0)$ and ascending powers of $\log\log(X - X_0)$ [19, 36] having the following shape:

$$\begin{aligned} Z \;=\; & \left(\frac{(n+1)(2n+1)}{n^2 B_0}\right)^{1/n} \left(-\,\frac{\log(X-X_0)}{X-X_0}\right)^{1/n} \\ & \times \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r,s}\,\frac{\big\{\log\big(-\log(X-X_0)\big)\big\}^s}{\big\{-\log(X-X_0)\big\}^r}\,. \end{aligned} \tag{2.27}$$

The parameters $X_0$ and $a_{1,0}$ are arbitrary constants of integration and

$$a_{0,0} \;=\; 1, \qquad a_{1,1} \;=\; -\,\frac{2n^2-1}{n(n+1)(2n+1)}\,.$$

The constant $a_{1,0}$ arises from the scaling freedom in Chazy's equation.

We claim that this series is a divergent asymptotic expansion uniformly valid as $X \to X_0$ in any sector on the Riemann surface of the form,

$$-\infty \;<\; \theta_1 \;\leq\; \arg(X - X_0) \;\leq\; \theta_2 \;<\; +\infty, \qquad \big|X - X_0\big| \;<\; \rho(\theta_1, \theta_2). \tag{2.28}$$

Furthermore, we claim that the series is the asymptotic expansion of the *general* integral of Chazy's equation because the third constant of integration multiplies a term that is exponentially small in comparison to the terms of the series and is therefore hidden beyond all orders.

We outline here a new method for analysing differential equations admitting formal series solutions of logarithms (logs) and logarithms of logarithms (loglogs). Several such equations appear below, and a generalization of Chazy's equation to higher orders appears in [19]. The first step of the method is to find a change of variables of the form,

$$c\log(X-X_0) \;=\; -\,t^{-1}+a\log t+K_2, \qquad Z^n \;=\; (X-X_0)^{-1}g(t), \tag{2.29}$$

where $c$ is a positive real number, $a$ is a real or complex number, and $K_2$ is a constant of integration related to the coefficient $a_{1,0}$, such that the transformed differential equation for the variable $g(t)$ admits a formal Laurent series solution in ascending powers of $t$, or a similar series involving fractional powers but no logarithms. The ascending series will always have an arbitrary coefficient that can be varied by varying $K_2$ in (2.29), and so this coefficient can always be set to zero without loss of generality.

In general, the series for $g(t)$ will be a zero-parameter divergent asymptotic series valid as $t \to 0$ in any sector properly contained in the right half complex $t$-plane, and may contain one or more parameters hidden beyond all orders if we restrict the domain to a smaller sector or a region having a cusp at $t=0$. In the original variables, the double series involving logs and loglogs will contain two manifest parameters ($X_0$ and $K_2$), and possibly further parameters beyond all orders, and will be a valid asymptotic expansion as $X \to X_0$ in a domain that covers infinitely many sheets of the Riemann surface for $Z(X)$ near the logarithmic branch point at $X = X_0$. Clearly, any differential equation for which this theory is applicable cannot have the Painlevé property.

Observe, first, that, under the conformal mapping (2.29), the positive real axis of the complex $t$-plane maps to a ray in the complex $X$-plane parallel to the positive real axis and having an endpoint at $X = X_0$. Secondly, all other rays in the right half of the complex $t$-plane map to spirals in the complex $X$-plane having $X_0$ as an asymptotic point, these spirals being asymptotic to equiangular spirals. Rays above and below the positive real axis map to spirals of opposite sense, and the region in the complex $t$-plane between such rays maps to a domain that runs over infinitely many sheets of a Riemann surface in the neighborhood of a logarithmic branch point at $X = X_0$. Such a domain contains sectors of the form (2.28) with $\theta_1$ and $\theta_2$ unrestricted (except for $\theta_2 > \theta_1$). More generally, a region of the form $|\mathrm{Im}\,t| \le (\mathrm{Re}\,t)^{\mu}$, where $1/2 < \mu < 2$, maps to a region contained between two spirals of opposite sense (not asymptotically equiangular unless $\mu = 1$) that contains sectors of the form (2.28). Such regions have a cusp at $t = 0$ when $\mu > 1$. On the other hand, sectors in the right half of the complex $t$-plane that do not contain the positive real axis map to regions on the Riemann surface contained between two spirals of the same sense having $X_0$ as an asymptotic point. Such regions do not contain sectors of the form (2.28). If the positive real axis is the boundary of a sector, the image is contained between a ray and a spiral and contains sectors with either $\theta_1 = 0$ or $\theta_2 = 0$.

In Chazy's equation (2.26), we can let $n$ be any positive real number. Change variables according to

$$(n+1)(2n+1)\log(X - X_0) \;=\; -\frac{n^2}{t} + (2n^2 - 1)\log t + K_2\,. \tag{2.30}$$

$$Z^n \;=\; (X - X_0)^{-1} g(t), \tag{2.31}$$

where $X_0$ and $K_2$ are constants of integration. The differential equation for $g(t)$ is easily constructed and has coefficients rational in $t$ and independent of $X_0$ and $K_2$. It admits a divergent asymptotic series solution in ascending powers,

$$g(t) \;\sim\; a_0 t^{-1} + a_1 + a_2 t + a_3 t^2 + a_4 t^3 + \ldots\,, \tag{2.32}$$

as $t \to 0$, where $a_0 = 1/B_0$ and $a_1$ is arbitrary. If the series is truncated at the term $a_m t^{m-1}$, then the error has magnitude $\mathrm{O}(t^m)$ as $t \to 0$. Without loss of generality, we can set $a_1 = 0$ because $a_1$ can be absorbed into the parameter $K_2$. In that case,

$$\begin{aligned}
a_2 \;&=\; -\frac{3}{n^2 B_0}\,(n+1)(2n+1),\\
a_3 \;&=\; \frac{1}{2n^5 B_0}\,(n+1)(2n+1)(36n^3 + 30n^2 + 7n + 3),\\
a_4 \;&=\; -\frac{1}{6n^7 B_0}\,(n+1)(2n+1)(872n^5 + 1704n^4 + 1330n^3\\
&\qquad + 594n^2 + 139n + 3).
\end{aligned}$$

To determine the domain of validity and the existence of any parameters hidden beyond all orders, the standard procedure [37, 38, 39, 40] is to examine first-order perturbations about any solution having the asymptotics (2.32). Let $g_0(t)$ be one such solution and let $g(t) = g_0(t) + \epsilon h(t)$, where squares and higher powers of $\epsilon$ will be neglected. Then $h(t)$ satisfies a third-order linear equation having an irregular singular point of rank 2 at $t = 0$. Two linearly independent solutions can be constructed by varying $X_0$ and $K_2$. These can be ignored. The third solution has the asymptotic form,

$$h(t) \;\sim\; K_3 t^\sigma \exp\{-\lambda/t^2 - \mu/t\}\Big\{1 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots\Big\}, \tag{2.33}$$

as $t \to 0$, where

$$\lambda = \frac{n^2}{2(n+1)(2n+1)}, \qquad \mu = \frac{6n^2+3n-1}{(n+1)(2n+1)},$$

$$\sigma = \frac{2n^2+3n+2}{(n+1)(2n+1)}, \qquad c_1 = \frac{(n+1)(2n^2+4n+3)}{2n^3}.$$

Now $h(t)$ is exponentially small in comparison to all the terms in the asymptotic expansion of $g_0(t)$ when $t \to 0$ from within the sector,

$$-\tfrac{1}{4}\pi + \delta \le \arg t \le \tfrac{1}{4}\pi - \delta, \qquad \delta > 0. \tag{2.34}$$

It follows that, within this sector, a one-parameter family of solutions of the nonlinear $g(t)$ equation are asymptotic to $g_0(t)$ to all orders. In the original variables, the corresponding solution has three integration constants, $X_0$ and $K_2$ appearing in the asymptotic expansion and $K_3$ appearing beyond all orders. The image of the sector under the map (2.30) is a domain between two spirals of opposite sense that covers infinitely many sheets of the Riemann surface for $Z(X)$ near the movable branch point $X = X_0$. For any real numbers $\theta_1$ and $\theta_2 > \theta_1$, there exists a positive real number $\rho(\theta_1, \theta_2)$ such that this domain contains the sector (2.28).

For $t$ in the right half-plane outside the sector (2.34), $g_0(t)$ continues to be a valid asymptotic expansion provided we suppress the hidden parameter $K_3$. Then $Z(X)$ becomes a nongeneric solution of Chazy's equation in a manner suggestive of the Painlevé-I functions that have a pole-free sector. Its asymptotic expansion (2.27) is valid in a region near $X = X_0$ that is larger than the aforementioned region contained between two spirals of opposite sense.

We now return to the Painlevé classification problem. In view of the fact that the exceptional case (2.25) cannot have the Painlevé property, we conclude that no equations of the form (2.18) can have the Painlevé property if any one of the degree bounds (2.19) is satisfied. It follows that all the degrees must satisfy

$$p \le 1, \qquad q \le 0, \qquad r \le 2, \qquad s \le 4, \tag{2.35}$$

or else one or more terms, but not all, are absent from (2.18). (It is convenient to treat the zero function as a polynomial having negative degree, the precise value being unimportant.) When equality holds in at least one of the inequalities (2.35), the Painlevé $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = y, \tag{2.36}$$

yields a reduced equation of the form,

$$y^{(\mathrm{iv})} = B(y)y'y''' + D(y)(y'')^2 + E(y)(y')^2y'' + F(y)(y')^4, \tag{2.37}$$

the prime denoting $d/dX$, where the coefficients are polynomials in $y$, any of which can be zero but not all can be zero simultaneously. A term is nonzero or zero according as equality or strict inequality holds in the corresponding member of (2.35).

In (2.37), rename $X$ to $x$ so that the prime denotes $d/dx$. Suppose the polynomials $B(y)$, $D(y)$, $E(y)$, and $F(y)$ have degrees $p_1$, $q_1$, $r_1$, and $s_1$, respectively. The $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = \alpha^{-1} y_0 + \alpha^{t-1} Y, \tag{2.38}$$

$$t = \max\Big\{p_1 + 2,\ q_1 + 2,\ \tfrac{1}{2}(r_1 + 3),\ \tfrac{1}{3}(s_1 + 4)\Big\},$$

yields a reduced equation of the same form with constant coefficients,

$$Y^{(\mathrm{iv})} = B_0 Y' Y''' + D_0 \big(Y''\big)^2 + E_0 \big(Y'\big)^2 Y'' + F_0 \big(Y'\big)^4, \tag{2.39}$$

where at least one coefficient on the right-hand side is nonzero. When we replace $Y'$ by $Z$, we arrive at the reduced equation associated with the Bureau symbol $P1$ derived from the Chazy polynomial class of third-order differential equations [6, 8, 19]. Here, it does not matter which ones have the Painlevé property. Every equation in this class, with one exception, admits an exact solution of the form,

$$Z = a_0 (X - X_0)^{-1}, \qquad Y = a_0 \log(X - X_0) + K, \tag{2.40}$$

where $X_0$ and $K$ are constants of integration, which is embedded in the general integral. The only exceptional case is the case $p = 1$ of (2.26) above, which is also known to admit movable logarithmic branch points. Hence every member of the class (2.39) fails to have the Painlevé property.

From the preceding paragraph, we conclude that equality is not permitted in any of the degree bounds (2.35). Hence the coefficients in the differential equation (2.18) are now restricted to the following forms:

$$\begin{aligned}
B(x,y,y') &= A(x,y), \\
D(x,y,y') &= 0, \\
E(x,y,y') &= B(x,y)y' + C(x,y), \\
F(x,y,y') &= D(x,y)(y')^3 + E(x,y)(y')^2 + F(x,y)y' + G(x,y),
\end{aligned}$$

where the coefficients $A(x,y)$, $\ldots$, $G(x,y)$ are polynomials in $y$. The differential equation itself becomes

$$\begin{aligned}
y^{(\mathrm{iv})} = {} & A(x,y)y''' + B(x,y)y'y'' + C(x,y)y'' + D(x,y)(y')^3 \\
& + E(x,y)(y')^2 + F(x,y)y' + G(x,y).
\end{aligned} \tag{2.41}$$

The differential equation is now in a form where a Bureau symbol [2, 3, 4] can be assigned, depending on the degrees of the polynomial coefficients in (2.41). Let the degrees in $y$ of $A(x,y)$, $\ldots$, $G(x,y)$ be $a$, $b$, $c$, $d$, $e$, $f$, and $g$, respectively. Let the zero polynomial be assigned the degree $-1$. Consider the Painlevé $\alpha$-test,

$$x = x_0 + \alpha X, \qquad y = \alpha^{-k} Y, \tag{2.42}$$

where $k$ is some positive rational number. Except when (2.41) is a linear differential equation, there will be a unique $k$ for which this $\alpha$-test will yield a reduced equation having $Y^{(\mathrm{iv})}$ on the left-hand side and at least one nonzero term on the right-hand side. In that case, the differential equation (2.41) will be assigned the Bureau symbol $Pk$. The value of $k$ is given by

$$k = \min\left\{\frac{1}{a}, \frac{1}{b+1}, \frac{2}{c}, \frac{1}{d+2}, \frac{2}{e+1}, \frac{3}{f}, \frac{4}{g-1}\right\}, \tag{2.43}$$

with the understanding that any fractions with zero or a negative number on the denominator count as $+\infty$ and so does $1/(d+2)$ when $D(x,y)$ is the zero polynomial. In the case of the linear differential equation, the Bureau symbol is $P\infty$ and the associated reduced equation is $Y^{(\mathrm{iv})} = 0$.

In the case of finite $k$, the reduced equation associated with Bureau symbol $Pk$ will take one of the following forms:

$$\begin{aligned} Y^{(\mathrm{iv})} &= A_0 Y^a Y''' + B_0 Y^{a-1} Y' Y'' + C_0 Y^{2a} Y'' + D_0 Y^{a-2} \big(Y'\big)^3 \\ &\qquad + E_0 Y^{2a-1} \big(Y'\big)^2 + F_0 Y^{3a} Y' + G_0 Y^{4a+1}, \end{aligned} \tag{2.44}$$

$$Y^{(\mathrm{iv})} = C_0 Y^c Y'' + E_0 Y^{c-1} \big(Y'\big)^2 + G_0 Y^{2c+1}, \tag{2.45}$$

$$Y^{(\mathrm{iv})} = F_0 Y^f Y', \tag{2.46}$$

$$Y^{(\mathrm{iv})} = G_0 Y^g. \tag{2.47}$$

The indices $k$ of these four equations are $1/a$, $2/c$, $3/f$, and $4/(g-1)$, respectively. The coefficients on the right-hand sides are constants, any of which can be zero provided the right-hand sides are nonzero. In cases where an exponent is negative, the corresponding coefficient is zero.

In the generic case of a differential equation having the Bureau symbol $Pk$, the associated reduced equation will have a particular integral of the form,

$$Y = a_0 \big(X - X_0\big)^{-k}, \tag{2.48}$$

which is embedded in the general integral. Hence the full equation will have either a general solution expressible as a power series of integer or non-integer powers of

$x - x_0$, possibly including powers of $\log(x - x_0)$, or a CFP-type perturbation expansion [9, 10]. Either way, the term $a_0(x - x_0)^{-k}$ will play the role of a leading term. All such equations will admit movable branch points in their general integrals when $k$ is not an integer. (In [41], we gave examples of Painlevé-type equations of the second order and second degree that admitted fractional Bureau symbols such as $P\frac{1}{2}$ and $P\frac{2}{3}$. In these cases, the general integrals were free of movable branch points, as required by the Painlevé property, but the singular integrals had the movable branch points that were indicated by the fractional Bureau symbols.)

Before restricting attention to integer Bureau symbols, we need to examine those exceptional cases where the reduced equations do not admit particular integrals of the form (2.48). Equations (2.46) and (2.47) do not have exceptional cases. Equation (2.45) has the exceptional case,

$$Y^{(\mathrm{iv})} = A_0\Big\{2Y^nY'' - (n+2)Y^{n-1}\big(Y'\big)^2\Big\}, \tag{2.49}$$

where $n \geq 1$ and $A_0 \neq 0$. This equation is qualitatively similar to Chazy's equation (2.26). It admits a 2-parameter solution embedded in the general integral that is expressible as a formal series of descending powers of $\log(X - X_0)$ and ascending powers of $\log\log(X - X_0)$. The leading term is

$$\begin{aligned} Y \sim{} & (\mathrm{const.})\big(X - X_0\big)^{-2/n}\big(\log(X - X_0)\big)^{1/n} \\ & \times\left\{1 + \mathrm{O}\left(\frac{\log\log(X - X_0)}{\log(X - X_0)}\right)\right\}, \end{aligned} \tag{2.50}$$

as $X \to X_0$. A family of equations containing both (2.26) and (2.49) was discussed in [19].

The method shown above for analysing (2.26) is applicable to this family. In the case of (2.49), use the change of variables,

$$\begin{aligned} & 2(n+1)(n+2)(3n+2)\log(X - X_0) \\ & \qquad = -n^3t^{-1} + (6n^3 + 11n^2 - 4)\log t + K_2\,, \end{aligned} \tag{2.51}$$

$$Y^n = (X - X_0)^{-2}g(t). \tag{2.52}$$

Then the differential equation for $g(t)$ admits an ascending asymptotic series of the same form as (2.32) above with $a_0 = -1/A_0$ and $a_1$ arbitrary. As before, we can set $a_1 = 0$ because $a_1$ can be absorbed into $K_2$. If we do not ask that the series be asymptotic to a solution having additional parameters hidden beyond all orders, then it is uniformly valid as $t \to 0$ from within any sector proiperly contained in the right half of the complex $t$-plane.

To investigate whether additional parameters are hidden beyond all orders, linearize the $g(t)$ equation about any particular solution having the ascending series as an asymptotic expansion. The linearized $g(t)$ equation has linearly independent solutions of the form,

$$h(t) \sim K_3 t^\sigma \exp\{-i\lambda/t^{3/2} - \mu/t - i\nu/t^{1/2}\}, \tag{2.53}$$

$$h(t) \sim K_4 t^\sigma \exp\{i\lambda/t^{3/2} - \mu/t + i\nu/t^{1/2}\}, \tag{2.54}$$

as $t \to 0$, where $\lambda$, $\mu$, $\nu$, and $\sigma$ are positive real numbers. The positive real axis is a Stokes line. Near this line, there is a domain $|\mathrm{Im}\, t| \leq (\text{const.})(\mathrm{Re}\, t)^{3/2}$ having a cusp at $t = 0$ where the factor $e^{-\mu/t}$ determines the asymptotics as $t \to 0$. In this domain, both forms of $h(t)$ are exponentially small in comparison to the terms of $g(t)$ and so the asymptotic expansion for $g(t)$ has two parameters hidden beyond all orders. In the original variables, the expansion has two constants appearing in the expansion involving logs and loglogs and two more constants beyond all orders. Despite the domain being restricted to the inside of a cusp in the $t$-plane, the image in the $X$ variable runs over infinitely many sheets of the Riemann surface for $Y(X)$ as $X \to X_0$ and contains sectors of the form (2.28). For $X$ corresponding to $t$ in either the first or fourth quadrant (but not both) of the complex $t$-plane, the asymptotic expansion defines a 3-parameter nongeneric solution having only one parameter hidden beyond all orders.

Equation (2.44) is exceptional when it takes the form,

$$\begin{aligned} Y^{(\mathrm{iv})} = {} & A_0\Big\{Y^n Y''' - (2n+1)Y^{n-1}Y'Y''\Big\} \\ & + B_0\Big\{Y^{n-1}Y'Y'' - (n+1)Y^{n-2}\big(Y'\big)^3\Big\} \\ & + C_0\Big\{Y^{2n}Y'' - (n+1)Y^{2n-1}\big(Y'\big)^2\Big\}, \end{aligned} \tag{2.55}$$

where $n \geq 1$ and $A_0$, $B_0$, and $C_0$ are constants, not all zero, $B_0$ being zero if $n = 1$. When $C_0 \neq 0$, equation (2.55) can be solved by a formal series in descending powers of $\{\log(X - X_0)\}^{1/2}$ and ascending powers of $\log\log(X - X_0)$. The leading term is

$$\begin{aligned} Y \sim {} & (\text{const.})\big(X - X_0\big)^{-1/n}\big(\log(X - X_0)\big)^{1/(2n)} \\ & \times \Big\{1 + \mathrm{O}\Big(\big(\log(X - X_0)\big)^{-1/2}\Big)\Big\}. \end{aligned} \tag{2.56}$$

The change of variables,

$$2(n+1)(2n+1)(3n+1)\log(X-X_0) \\ = \; -n^3t^{-1} + (12n^3+11n^2-1)\log t + K_2\,, \tag{2.57}$$

$$Y^n \;=\; (X-X_0)^{-1}g(t), \tag{2.58}$$

$\mathrm{Re}\, t > 0$, yields a differential equation for $g(t)$ admitting an asymptotic solution of the form,

$$g(t) \;\sim\; a_0t^{-1/2} + a_1 + a_2t^{1/2} + a_3t + a_4t^{3/2} + \;\ldots\;, \tag{2.59}$$

with

$$a_0 \;=\; \pm\big(-C_0\big)^{-1/2}, \qquad a_1 \;=\; \frac{2(n+1)A_0 + B_0}{nC_0}\,, \qquad a_2 \text{ arbitrary.}$$

When $(A_0)^2 + 4C_0 \neq 0$, the linearized $g(t)$ equation has two linearly independent solutions of the form,

$$h(t) \;\sim\; K_3t^{\sigma}\exp\big\{-\lambda/t^{3/2} - \mu/t - \nu/t^{1/2}\big\}, \tag{2.60}$$

where

$$\lambda \;=\; \frac{n^3a_0\big(A_0 \pm \sqrt{(A_0)^2+4C_0}\,\big)}{6(n+1)(2n+1)(3n+1)}\,. \tag{2.61}$$

When $(A_0)^2 + 4C_0 = 0$, the two solutions take the alternative form,

$$h(t) \;\sim\; K_3t^{\sigma}\exp\big\{-\lambda/t^{3/2} \pm \mu/t^{5/4} + \nu/t + \;\ldots\big\}, \tag{2.62}$$

and this leading term multiplies an asymptotic series in ascending powers of $t^{1/4}$.

The positions of the Stokes lines depend on the ratios $B_0/A_0$ and $C_0/(A_0)^2$ and the sign of $a_0$, but there are always sectors in the right-half $t$-plane where at least one of the forms taken by $h(t)$ is exponentially small in comparison to $g(t)$. Hence, as in the previous cases, the asymptotic series for $Y(X)$ contains two integration constants arising from the scaling and translation freedom in (2.55) and at least one additional integration constant hidden beyond all orders when $X \to X_0$ along appropriate radial or spiral paths. The expansion is uniformly valid as $X \to X_0$ in a sector of the form (2.28) on the Riemann surface of the variable $X$ for at least a two-parameter solution of equation (2.55). In that case, the number of parameters hidden beyond all orders is equal to the number of solutions $h(t)$ of the linearized equation that are recessive along the positive real axis of the complex $t$-plane.

When $C_0 = 0$ and $B_0 \neq -2(n+1)A_0$, the differential equation (2.55) admits a solution with leading term,

$$Y \sim (\text{const.})\big(X - X_0\big)^{-1/n}\big(\log(X - X_0)\big)^{1/n} \times \left\{1 + \mathrm{O}\left(\frac{\log\log(X - X_0)}{\log(X - X_0)}\right)\right\}. \tag{2.63}$$

This case includes the derivative of equation (2.26). When $B_0 = -2(n+1)A_0$, equation (2.55) with $C_0 = 0$ admits a solution with leading term,

$$Y \sim (\text{const.})\big(X - X_0\big)^{-1/n}\big(\log(X - X_0)\big)^{2/n} \times \left\{1 + \mathrm{O}\left(\frac{\log\log(X - X_0)}{\log(X - X_0)}\right)\right\}. \tag{2.64}$$

The analysis of these cases is similar to the analysis of (2.26), (2.49), and (2.55) with $C_0 \neq 0$. The leading term behaviors of $g(t)$ and $h(t)$ are different in each case but can be found by straightforward methods. In these cases, it is found that the asymptotic solutions (2.63) and (2.64) are uniformly valid as $X \to X_0$ in a sector on the Riemann surface of the form (2.28) and may or may not contain hidden constants of integration beyond all orders. If there are no hidden integration constants when $X \to X_0$ in a sector, then we can always capture at least one by letting $X \to X_0$ along paths contained between two suitable spirals with $X_0$ as asymptotic point. In any case, we can conclude that none of the exceptional cases have the Painlevé property.

The conclusion reached so far is that equations of the form (2.1) in the polynomial class having the Painlevé property must take the form (2.41) with the degrees of the coefficients restricted in such a way that the Bureau symbol is $Pk$, where $k$ is a positive integer or $\infty$. Because the exponents in (2.44)–(2.47) must be nonnegative integers, the only admissible Bureau symbols are

$$P1, \quad P2, \quad P3, \quad P4, \quad P\infty. \tag{2.65}$$

After suitably renaming parameters, we can write the corresponding reduced equations associated with these Bureau symbols in the forms,

$$P1: \qquad Y^{(\mathrm{iv})} = AYY''' + BY'Y'' + CY^2Y'' + DY\big(Y'\big)^2 + EY^3Y' + FY^5, \tag{2.66}$$

$$P2: \qquad Y^{(\mathrm{iv})} = AYY'' + B\big(Y'\big)^2 + CY^3, \tag{2.67}$$

$$P3: \qquad Y^{(\mathrm{iv})} = AYY', \tag{2.68}$$

$$P4: \qquad Y^{(\mathrm{iv})} = AY^2, \tag{2.69}$$

$$P\infty: \qquad Y^{(\mathrm{iv})} = 0. \tag{2.70}$$

In Paper I, we carried out the Painlevé analysis of all of these cases except (2.66). We found that there we no equations of the type considered having symbols $P3$ or $P4$ because the resonance numbers were not integers. The general fourth-order linear equation is the only equation having symbol $P\infty$. (Note that nonlinear equations of higher degree can have symbol $P\infty$, e.g., SD-VI in [36] and several equations in [41].) In the case of Bureau symbol $P2$, we found a complete list of five Painlevé-type equations to which we added the Bureau barrier equation. These are the equations denoted F-I, F-II, ..., F-VI and given above by (1.4)–(1.9).

To complete the Painlevé classification of the fourth-order polynomial class, we need to find all reduced equations of the form (2.66) having the Painlevé property and thence all full equations that are built upon these reduced equations. The full equations necessarily take the form (1.2) above.

## 3. Equations with $E = F = 0$

In this section, we consider the problem of constructing all reduced equations of the form (1.1) subject to the restriction,

$$E = 0, \qquad F = 0.$$

The construction of the corresponding full equations of the form (1.2) will be postponed till Section 6. This subcase is the one that has been treated most fully in the literature, especially by Bureau [8] and Martynov [21]. We can therefore review this subcase quickly, but there are some gaps in the classical work that need to be closed. We will prove that the only Painlevé-type reduced equations of the type considered, up to a scaling, are the reduced versions of equations F-VII, F-VIII, and F-IX given by (1.13)–(1.15) above.

We will examine the following three reduced equations in turn:

$$y^{(\mathrm{iv})} = Ayy''' + By'y'', \tag{3.1}$$

$$y^{(\mathrm{iv})} = Ayy''' + By'y'' + C\{y^2y'' - 2y(y')^2\}, \tag{3.2}$$

$$y^{(\mathrm{iv})} = Ayy''' + By'y'' + Cy^2y'' + Dy(y')^2. \tag{3.3}$$

All coefficients are constant. In (3.1), $A$ and $B$ are not both zero; in (3.2), $C \neq 0$; and in (3.3), $D \neq -2C$.

Equation (3.1) has the first integral,

$$y''' = Ayy'' + \tfrac{1}{2}(B-A)(y')^2 + K, \tag{3.4}$$

where $K$ is a constant of integration. This equation belongs to the Chazy polynomial class [6, 8, 19]. Without the additive constant $K$, the only Painlevé-type equations of the form (3.4), up to trivial scalings, are known to be

$$\text{Chazy-I}: \qquad y''' = -6(y')^2, \qquad \{1,6\}, \tag{3.5}$$

$$\text{Chazy-II}: \qquad y''' = -2yy'' - 2(y')^2, \qquad \{2,3\}, \tag{3.6}$$

$$\text{Chazy-III}: \qquad y''' = 2yy'' - 3(y')^2, \qquad \{-2,-3\}. \tag{3.7}$$

The numbers in braces are the resonance numbers, excluding $-1$. Hence the first task is to determine which of these equations continues to have the Painlevé property after a constant is added to the right-hand side. It is easy to see that (3.5) and (3.6) can support an additive constant because these equations are easily integrated. Hence we have arrived at the reduced forms of equations F-VII and F-VIII:

$$\text{F-VII}: \qquad y^{(\text{iv})} = -12y'y'', \qquad \{1,4,6\}, \tag{3.8}$$

$$\text{F-VIII}: \qquad y^{(\text{iv})} = -2yy''' - 6y'y'', \qquad \{2,3,4\}. \tag{3.9}$$

The corresponding full equations are determined in Section 7.

The Chazy-III equation is one of Chazy's celebrated natural barrier equations, the other being Chazy-XII. The problem of deciding whether or not Chazy-III can support recessive terms, including the particular case of an additive constant, was left open by the classical authors. Pickering [9], with the perturbative Painlevé test which we call the CFP test [10], gave the first proof that there are no Painlevé-type equations of the form,

$$\begin{aligned} y''' = {} & 2yy'' - 3(y')^2 + A(x)y'' + B(x)yy' \\ & + C(x)y^3 + D(x)y' + E(x)y^2 + F(x)y + G(x), \end{aligned} \tag{3.10}$$

except those that can be obtained from the reduced equation (3.7) by a gauge transformation. By a choice of gauge, one can set $A(x) = 0$ and $D(x) = 0$. In an earlier draft of the CFP test, where only the first perturbation was examined, Fordy and Pickering [42] were able to prove that $B(x)$, $C(x)$, and $E(x)$ must vanish. Pickering [9] and Conte et al. [10] later showed that the second perturbation forces $F(x)$ and $G(x)$ to vanish also. In particular, these authors have proved that the equation,

$$y''' = 2yy'' - 3(y')^2 + K, \tag{3.11}$$

does not have the Painlevé property. Hence the fourth-order equation,

$$y^{(\mathrm{iv})} \;=\; 2yy''' - 4y'y'', \qquad \{-2,-3,4\}, \tag{3.12}$$

obtained by differentiating the Chazy-III equation, does not have the Painlevé property. The proof for this case is contained in the analysis of equation (5.15) below.

An alternative proof that we find attractive is Painlevé's original method of $\alpha$-expansions [26, pp. 224–225] (reviewed by Ince [27, pp. 329–330]). It deals with the problem of recessive terms for Chazy-III and XII quite efficiently. We show the method for the special case (3.11). The Painlevé $\alpha$-test,

$$x \;=\; x_0 + \alpha X, \qquad y \;=\; \alpha^{-1} Y + \alpha^3 Y_1, \tag{3.13}$$

forces $Y(X)$ to satisfy the pure Chazy-III equation while $Y_1(X)$ satisfies the linear equation,

$$Y_1''' - 2YY_1'' + 6Y'Y_1' - 2Y''Y_1 \;=\; K. \tag{3.14}$$

Let $Y(X)$ be the exact solution,

$$Y(X) \;=\; -6\big(X - X_0\big)^{-1} + C\big(X - X_0\big)^{-2}, \tag{3.15}$$

where $X_0$ and $C$ are constants. Then (3.14) has the general solution,

$$\begin{aligned} Y_1(X) \;=\; & \frac{e^{-2C/(X-X_0)}}{(X-X_0)^2}\left\{\tfrac{1}{42}K\int (X-X_0)^4 e^{2C/(X-X_0)}\,dX + K_2\right\} \\ & + \frac{K_3 X + K_4}{(X-X_0)^3}. \end{aligned} \tag{3.16}$$

When $K \neq 0$, the essential singularity at $X = X_0$ is logarithmically branched. This concludes the proof.

The class of equations (3.1) includes the equation,

$$y^{(\mathrm{iv})} \;=\; yy''' - 3y'y'', \tag{3.17}$$

whose first integral is

$$y''' \;=\; yy'' - 2(y')^2 + K. \tag{3.18}$$

Equation (3.17) is included among the exceptional equations (2.55) above and equation (3.18) without the additive constant $K$ is the case $n = 1$ of Chazy's equation (2.26). These equations admit asymptotic expansions involving logs and loglogs with two parameters apparent in the expansions and two more parameters hidden beyond all orders. Hence these equations do not have the Painlevé property.

The class of equations (3.2) is a class that was given special attention by Martynov [21]. It includes the particular case,

$$y^{(\mathrm{iv})} = A_0\{yy''' - 3y'y''\} + B_0\{y^2y'' - 2y(y')^2\}, \tag{3.19}$$

which was recognized as problematic by Chazy [6], who proved that its solutions do not admit poles or algebraic branch points. This equation is a special case of the exceptional equation (2.55) and therefore admits movable logarithmic branch points.

Martynov observed that exactly eight equations in the class (3.2) have integer resonance numbers that obey the standard rules. These equations, with their resonance numbers in braces, are

$$(i) \qquad y^{(\mathrm{iv})} = 4yy''' - 24y'y'' + 12\{y^2y'' - 2y(y')^2\}, \qquad \{1, 2, 12\}, \tag{3.20}$$

$$(ii) \qquad y^{(\mathrm{iv})} = yy''' - 15y'y'' + 3\{y^2y'' - 2y(y')^2\}, \qquad \{1, 3, 8\}, \tag{3.21}$$

$$\begin{aligned}(iii) \qquad y^{(\mathrm{iv})} = &-24yy''' + 60y'y'' \\ &- 72\{y^2y'' - 2y(y')^2\}, \qquad \{1, -2, -12\},\end{aligned} \tag{3.22}$$

$$\begin{aligned}(iv) \qquad y^{(\mathrm{iv})} = &-21yy''' + 51y'y'' \\ &- 63\{y^2y'' - 2y(y')^2\}, \qquad \{1, -3, -8\},\end{aligned} \tag{3.23}$$

$$\begin{aligned}(v) \qquad y^{(\mathrm{iv})} = &-20yy''' + 48y'y'' \\ &- 60\{y^2y'' - 2y(y')^2\}, \qquad \{1, -4, -6\},\end{aligned} \tag{3.24}$$

$$\begin{aligned}(vi) \qquad y^{(\mathrm{iv})} = &-17yy''' + 39y'y'' \\ &- 30\{y^2y'' - 2y(y')^2\}, \qquad \{2, -2, -6\},\end{aligned} \tag{3.25}$$

$$\begin{aligned}(vii) \qquad y^{(\mathrm{iv})} = &-16yy''' + 36y'y'' \\ &- 28\{y^2y'' - 2y(y')^2\}, \qquad \{2, -3, -4\},\end{aligned} \tag{3.26}$$

$$\begin{aligned}(viii) \qquad y^{(\mathrm{iv})} = &-14yy''' + 30y'y'' \\ &- 12\{y^2y'' - 2y(y')^2\}, \qquad \{3, -2, -4\}.\end{aligned} \tag{3.27}$$

(Martynov's list has 11 equations because he included (3.8), (3.9), and (3.12), where $C = 0$; we also corrected a misprint in (*iv*).) Martynov claimed that all of these equations violate the Painlevé property, but his proof does not appear to be complete. In the case of $(i)$ and $(ii)$, he knew that the standard ascending series solution,

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n-1}, \tag{3.28}$$

must be augmented by logarithm terms starting at the term where $n$ coincides with the second positive resonance. In the case of $(iii)$–$(viii)$, he claimed that logarithms appear in the descending series about the fixed singularity at $x = \infty$. Certainly, if the point at infinity is a branch point, then there must also be a finite branch point somewhere, and this latter point will be movable because the equations are autonomous (i.e., have constant coefficients). However, we find that logarithms appear in the descending series only in the cases $(iii)$, $(vi)$, and $(viii)$. In the other three cases, the descending series converges to a 3-parameter solution analytic outside a bounded region.

The best test to use is the CFP test [9, 10, 42]. It gives a violation at the first perturbation order in each of the cases $(iii)$–$(viii)$. Furthermore, the test gives an answer quickly without taking into account any inconvenient resonance numbers such as $-12$ or $-8$. Equations (3.8), (3.9), (3.12), and (3.20)–(3.27) are special cases of equations (5.5)–(5.15) below, which will also be analysed by CFP tests.

An alternative approach to the class (3.2) is by the algorithm introduced in the preceding section. These equations all admit asymptotic expansions in descending powers of $\log(x - x_0)$ and ascending powers of $\log\log(x - x_0)$ and so do not have the Painlevé property. When $B = -3A$, the equation becomes (3.19), which has already been examined. When $B \neq -3A$, the change of variables,

$$\begin{aligned} &2(3A+B)^2 \log(x - x_0) \\ &\qquad = \; -(3A+B)^2 t^{-1} + \big\{(3A+B)(A-B) + 12C\big\} \log t + K_2 \,, \end{aligned} \tag{3.29}$$

$$y = (x - x_0)^{-1} g(t), \tag{3.30}$$

$\mathrm{Re}\, t > 0$, yields a differential equation for $g(t)$ admitting an asymptotic solution of the form (2.32) with $a_0 = -(3A+B)/C$ and $a_1$ arbitrary. We can set $a_1 = 0$ because $a_1$ can be absorbed into the parameter $K_2$. When $A^2 + 4C \neq 0$, the linearized $g(t)$ equation has two linearly independent solutions with asymptotic behavior,

$$h(t) \sim K_3 t^{\sigma} \exp\big\{-\lambda/t^2 - \mu/t\big\}, \tag{3.31}$$

where

$$\lambda = \frac{3A+B}{8C} \big\{-A \pm \sqrt{A^2 + 4C}\,\big\}. \tag{3.32}$$

When $A^2 + 4C = 0$, the two solutions take the alternative forms,

$$h(t) \sim K_3 t^\sigma \exp\bigl\{-\lambda/t^2 \pm \mu/t^{3/2} + \nu/t \pm \rho/t^{1/2}\bigr\}, \tag{3.33}$$

and this leading term multiplies an asymptotic series in ascending powers of $t^{1/2}$.

The Stokes lines depend on the ratios $B/A$ and $C/A^2$. In the original variables, the series involving logs and loglogs will be a valid asymptotic expansion as $x \to x_0$ in any sector on the Riemann surface with vertex at the branch point $x_0$. Two integration constants are manifest in this expansion. There may or may not be additional parameters hidden beyond all orders, depending on whether zero, one, or both values of $\lambda$ have positive real part. In any case, there are sectors in the right half $t$-plane where at least one of the forms of $h(t)$ is recessive and hence a third integration constant can be captured by taking the limit $x \to x_0$ along suitable radial or spiral paths on the Riemann surface.

While none of the equations (3.2) have the Painlevé property, it remains to be seen whether any particular cases, especially the Martynov equations (3.20)–(3.27), are integrable or partially integrable in some sense or are otherwise worthy of further study by Painlevé analysts. We take the view that certain non-Painlevé equations should be studied carefully to see what can be learned about their singularity structure and its possible relation to the deep question of integrability [23]. The discussion of the Chazy problematic equation (2.26) and its relatives in the previous section suggests that these equations may be of continuing interest. Other non-Painlevé equations that have attracted our attention are the Bureau barrier equation (1.5) [5] and the Clarkson-Olver equation (1.25).

Equation (3.3), in which $2C + D \neq 0$, can be handled by standard methods [8]. We sketch those methods here as an introduction to the next three sections, where the calculations are more intricate. If we substitute a Laurent series of the form (3.28), we find that the leading coefficient $a_0$ takes two values given by the quadratic equation,

$$(2C + D)(a_0)^2 - 2(3A + B)a_0 - 24 \;=\; 0. \tag{3.34}$$

Let the roots be denoted $a_{01}$ and $a_{02}$.

For each choice of $a_0$, the resonance numbers are the numbers $n_1$, $n_2$, and $n_3$ such that the recurrence relation for the $a_n$ in the series expansion (3.28) takes the form,

$$(n+1)(n-n_1)(n-n_2)(n-n_3)a_n \;=\; \phi_n\bigl(a_0, a_1, \ldots, a_{n-1}\bigr). \tag{3.35}$$

By a standard calculation, it is found that these numbers are the roots of the cubic equation,

$$\begin{aligned} n^3 - 11n^2 + 46n - 48 - Aa_0(n-1)(n-6)& \\ + Ba_0(n-2) - C(a_0)^2 n \;=\; 0.& \end{aligned} \tag{3.36}$$

According to standard Painlevé tests [8, 10, 21], the (nontrivial) resonance numbers associated with each leading coefficient $a_0$ must be distinct integers and may not take the values 0 or $-1$. Also the roots of the $a_0$ equation must be distinct. These rules apply to single differential equations of the first degree and semilinear partial differential equations in the polynomial class of any order where a Bureau symbol of $Pk$ type is defined.

Let the triad $\{n_1, n_2, n_3\}$ denote the resonance numbers associated with the case $a_0 = a_{01}$ and let $\{m_1, m_2, m_3\}$ denote the resonance numbers associated with $a_0 = a_{02}$. Define the products,

$$N_1 := n_1 n_2 n_3 \;=\; 2(3A+B)a_{01} + 48 \;=\; 24\big(a_{02} - a_{01}\big)/a_{02}\,, \tag{3.37}$$

$$N_2 := m_1 m_2 m_3 \;=\; 2(3A+B)a_{02} + 48 \;=\; 24\big(a_{01} - a_{02}\big)/a_{01}\,. \tag{3.38}$$

These products are never 0, $\pm 1$, $\pm p$, or $p^2$, where $p$ is a prime number. From the expressions in terms of $a_{01}$ and $a_{02}$, we see that $N_1$ and $N_2$ satisfy the Diophantine equation,

$$\frac{1}{N_1} + \frac{1}{N_2} = \frac{1}{24}\,, \tag{3.39}$$

or its equivalent rearrangement (if the zero solution is excluded),

$$(N_1 - 24)(N_2 - 24) \;=\; 576. \tag{3.40}$$

This equation has 21 unordered solutions.

Now $N_1 - 24$ is a divisor of 576. Without loss of generality, we can make it the divisor that lies between $-24$ and $+24$, inclusive of the upper limit. This restricts $N_1$ to the 21 values,

$$6,\; 8,\; 12,\; 15,\; 16,\; 18,\; 20,\; 21,\; 22,\; 23,$$

$$25,\; 26,\; 27,\; 28,\; 30,\; 32,\; 33,\; 36,\; 40,\; 42,\; 48,$$

from which we can discard $N_1 = 23$ and $N_1 = 25$ because these values do not have legal factorizations into resonance numbers. Next, order the factors of $N_1$ so that $|n_1| \le |n_2| \le |n_3|$. Now $N_1 \le 48$ implies that $|n_1| \le 3$, with equality being possible only in the cases $N_1 = 36$ and $N_1 = 48$. A straightforward count shows that there are a total of 106 permissible factorizations of $N_1$ into resonance numbers. In each case, we can construct the differential equation uniquely up to a scaling factor, which we may take to be the parameter $a_{01}$. The coefficients are given by

$$\begin{aligned} Aa_{01} &= n_1 + n_2 + n_3 - 11, && (3.41) \\ Ba_{01} &= \tfrac{1}{2}(N_1 - 48) - 3Aa_{01}\,, && (3.42) \\ C(a_{01})^2 &= (7A + B)a_{01} + 46 - (n_1 n_2 + n_1 n_3 + n_2 n_3), && (3.43) \\ D(a_{01})^2 &= N_1 - 24 - 2C(a_{01})^2. && (3.44) \end{aligned}$$

We can assign $a_{01}$ any nonzero value because of the trivial scaling freedom in the differential equation (3.3). Let us scale the equation so that

$$a_{01} = 1, \qquad a_{02} = -\frac{24}{2C + D} = \frac{24}{24 - N_1}\,, \tag{3.45}$$

the second equation being a consequence of (3.34) and (3.44). With this choice, $A$ and $D$ are always integers while $B$ and $C$ are integers when $N_1$ is even and half-odd-integers when $N_1$ is odd.

So far, we have a list of 106 differential equations of the form (3.3) in which three of the six resonance numbers are integers. The remaining three resonance numbers $\{m_1, m_2, m_3\}$ associated with the leading coefficient $a_{02}$ are also required to be integers. So far, only their product $N_2$ is known to be an integer. In each case, we must factorize the cubic equation (3.36) with $m$ replacing $n$ and $a_0 = a_{02}$. We find that 78 of the 106 cubics are irreducible over the rational numbers, 26 factorize into a linear and a quadratic factor, and only two factorize into three linear factors. Alternatively, we can discard 12 cubics before factorizing them because the sum of their roots is not an integer. Of the 94 remaining, 68 are irreducible, 24 have a single linear factor, and two have three linear factors.

We see that only two differential equations in the class (3.3) survive the condition that both sets of resonance numbers be integers subject to the usual rules. This result was obtained by Bureau [8] and Martynov [21]. The first of these equations is the reduced form of equation F-IX, to which we have appended its resonance numbers:

$$\text{F-IX}: \qquad y^{(\mathrm{iv})} = -3yy''' - 9y'y'' - 3y^2y'' - 6y(y')^2, \tag{3.46}$$

$$a_{01} = 1: \ \{1, 3, 4\}, \qquad a_{02} = 2: \ \{3, 4, -2\}. \tag{3.47}$$

A first integral is

$$y''' = -3yy'' - 3(y')^2 - 3y^2y' + K_1, \tag{3.48}$$

which is a special case of the Chazy-IV equation [6, 8, 19]. Hence the reduced form of equation F-IX has the Painlevé property. The corresponding full equation is determined in Section 7.

The second of the two surviving equations is

$$y^{(\mathrm{iv})} = -yy''' - 11y'y'' - y^2y'' - 2y(y')^2, \tag{3.49}$$

$$a_{01} = 1: \quad \{1, 4, 5\}, \qquad a_{02} = 6: \quad \{4, 6, -5\}. \tag{3.50}$$

A first integral is

$$y''' = -yy'' - 5(y')^2 - y^2y' + K_1, \tag{3.51}$$

which is the reduced Chazy-VI equation [6, 8, 19] plus an additive constant. It is known that an additive constant is not a permissible recessive term in the Chazy-VI equation. In fact, the general solution of (3.49) and (3.51) has the series expansion,

$$\begin{aligned} y = {} & \frac{1}{x - x_0} + a_1 - \tfrac{1}{9}(a_1)^2(x - x_0) - \tfrac{1}{405}\big((a_1)^4 + 27K_1\big)(x - x_0)^3 \\ & + \big\{a_5 + \tfrac{1}{36}a_1K_1\log(x - x_0)\big\}(x - x_0)^4 + \dots, \end{aligned} \tag{3.52}$$

in which a movable logarithmic branch point is evident.

In this section, we found that three and only three reduced equations of the forms (3.1)–(3.3) have the Painlevé property. These are the reduced versions of F-VII, F-VIII, and F-IX given, respectively, by (3.8), (3.9), and (3.46). The corresponding full equations and their solutions were first given by Bureau [8]. These results, including an improved presentation of F-VII, are included in Section 7 below.

Observe that all reduced equations of the forms (3.1) and (3.3) that had integer resonances were derivatives of third-order Chazy equations. The derivatives of Chazy Classes I, II, and IV have the Painlevé property while the derivatives of III and VI do not. The derivatives of Chazy Classes V, VII, and VIII also take the form (3.3). These all have the resonance number 4. When differentiated, any third-order equation having Bureau symbol $P1$ picks up an additional resonance number 4. Hence the derivatives of Chazy Classes V, VII, and VIII have repeated resonance numbers and therefore do not have the Painlevé property. These results were known to Bureau [8] and Martynov [21]. In the next three sections, the derivatives of Chazy Classes IX, XI, and XII will appear, but the derivative of Chazy Class X will not appear because it also has the repeated resonance number 4. The derivative of Chazy Class XIII, namely F-I, has Bureau symbol $P2$. This case, which was also studied by Bureau [8], made its appearance in paper I.

## 4. Discrete solutions of the Diophantine equations

In this and the next two sections, we carry out the construction of all reduced equations of the form,

$$y^{(\mathrm{iv})} = Ayy''' + By'y'' + Cy^2y'' + Dy(y')^2 + Ey^3y' + Fy^5, \tag{4.1}$$

in which the coefficients $E$ and $F$ are not both zero. The construction of the corresponding full equations of the form (1.2) will be postponed till Section 7. An incomplete treatment of these cases was given by Martynov [21] and Exton [25].

In this section, we obtain the principal Diophantine equations that provide necessary conditions for equation (4.1) to have integer resonances and we carry out the construction of the admissible reduced equations in all but two nongeneric situations. The latter cases are postponed to the next two sections. Some of the methods in this section are what may be called the "traditional" methods, namely, methods based entirely on obtaining conditions for integer resonances through a series of nested Diophantine equations. These methods are demonstrated by Bureau [8] in his derivation of the third-order Chazy equations I–XIII. These are also the methods we believe were used by Exton [25] and Martynov [21] on the corresponding fourth-order problem, but those authors were not able to complete the analysis with the computing power available to them in the early 1970s.

The traditional methods are unable to deal with the nongeneric cases treated in the next section, where the Diophantine equations have infinite families of solutions. In these cases, the CFP test plays a vital role, together with standard methods for removing logarithms at positive resonances. In Section 6, we treat the subcase $A = 0$, which requires special handling, by a mixture of positive resonance tests and traditional methods.

We will prove that the only Painlevé-type reduced equations of the type considered, up to a scaling, are the reduced versions of equations F-X/XI and F-XII–F-XVIII given above by (1.16)–(1.24), the cases F-X and F-XI being combined here because they have the same reduced equation. Most of these equations appear in the next two sections. Under the hypotheses of this section, we will prove that the only Painlevé-type equations are F-XV and F-XVI and that the only other equation having admissible integer resonances is the non-Painlevé equation (4.6) (or, equivalently, (4.7)) below.

Both the full and reduced forms of equation F-XV are new. This equation is a group-invariant reduction of a new Painlevé-type partial differential equation,

$$u_{xxxt} = -2uu_{xxt} - 6u_xu_{xt} - 5u_tu_{xx} - 2u^2u_{xt} - 7uu_xu_t - u^3u_t\,, \tag{4.2}$$

where the dependent variable is $u(x,t)$ and subscripts denote partial differentiation. Equation (4.2) is a generalization of the Chazy-VI partial differential equation,

$$u_{xxt} = -uu_{xt} - 5u_x u_t - u^2 u_t\,, \tag{4.3}$$

which is also new. The Painlevé classification problems that yielded these two equations are the subjects of a paper in preparation.

Equation F-XVIII was also missed in earlier Painlevé classification attempts but is now well known to be a group-invariant reduction [43] of the Modified Sawada-Kotera equation,

$$\begin{aligned} u_{xxxxx} = {} & -5(u_x - u^2)u_{xxx} - 5(u_{xx})^2 \\ & + 20uu_x u_{xx} + 5(u_x)^3 - 5u^4 u_x + u_t\,. \end{aligned} \tag{4.4}$$

found by Fordy and Gibbons [20], who observed that this equation is also the modified version of the Kaup-Kupershmidt equation.

Exton and Martynov obtained the reduced forms of F-XI–F-XIV, and F-XVI. However, Exton rejected F-XIV, which is the derivative of Chazy-IX, because he gave an incorrect proof that Chazy-IX was not a Painlevé-type equation, a proof which he claimed also applied to Chazy-X. Both authors found the full versions of F-XII and F-XIII, gave incomplete recessive terms for F-XI, and missed the companion equation F-X. Exton found the full version of F-XVI and knew that it was linearizable. Martynov also found the reduced form and an incomplete full form of equation F-XVII, which is a group-invariant reduction of the Modified KdV5 equation,

$$\begin{aligned} u_{xxxxx} = {} & 10\big\{u^2 - 2R(t)\big\}u_{xxx} + 40uu_x u_{xx} + 10(u_x)^3 \\ & - 30\big\{u^2 - 2R(t)\big\}^2 u_x + u_t\,, \end{aligned} \tag{4.5}$$

where $R(t)$ is an arbitrary locally analytic function of $t$.

The methods employed in this section also reveal the new differential equation,

$$y^{(\mathrm{iv})} = 2yy''' - 17y'y'' + 11y^2y'' - 15y(y')^2 + 4y^3y' - y^5. \tag{4.6}$$

This equation has four sets of resonance numbers, all of which are integers. However, like equation (3.49) above, it fails the freedom-from-logarithms test at one of these resonances. Nevertheless, equation (4.6) may be of sufficient interest to warrant adding it to the list in Section 1 and giving it a number (e.g., F-XX).

After replacing $y$ by $y/2$, equation (4.6) can be written,

$$\left(\frac{d}{dx} + y\right)\left\{y''' - 2yy'' + 3(y')^2 + \tfrac{1}{16}\big(6y' - y^2\big)^2\right\} = 0, \tag{4.7}$$

which we choose to be the standard form for this equation. Setting the quantity in braces to zero gives the Chazy-XII equation with $N = 10$, which can be solved by inversion of a hypergeometric function [6]. It follows that equation (4.6) admits a three-parameter solution that is meromorphic on one side of a circular natural barrier. Interestingly, this same Chazy-XII equation is also related to another near-miss Painlevé equation, namely, the sixth-order member of the Bureau barrier equation hierarchy [1, 5],

$$y^{(\mathrm{vi})} = 2yy^{(\mathrm{iv})} - 16y'y''' + 15(y'')^2. \tag{4.8}$$

If the right-hand side of (4.8) is set to zero, the function $w := 2y'/y$ satisfies the Chazy-XII equation with $N = 10$.

Equation (4.7) is not to be confused with the integrable Clarkson-Olver equation [22, 19], which takes the similar form,

$$\text{F-XIX}: \qquad \left(\frac{d}{dx} - \tfrac{4}{3}y\right)\left\{y''' - 2yy'' + 3(y')^2 - \alpha\big(6y' - y^2\big)^2\right\} = 0. \tag{4.9}$$

This equation, to which we have assigned a number, should be of special interest to those researchers that have been attracted to the Chazy natural barrier equations and the related differential equations defining the Schwarz triangle functions and Fuchsian/Kleinian functions. The singularity structure of F-XIX is, in general, very complicated. The only case having the Painlevé property is the special case $\alpha = 4/27$, which is equivalent under scaling to the reduced form of equation F-XVI. The general solution of F-XIX in terms of a linear differential equation is included in Section 7.

In this section, the problem will be divided into two natural subcases: (a) $F = 0$ and $E \neq 0$, and (b) $F \neq 0$. As already mentioned, there are two nongeneric cases that are postponed to the next two sections. Section 5 will discuss the cases with $N_1 = 24$, where $N_1$ is a product of resonance numbers and appears in the Diophantine equations (4.18) and (4.51) below. Section 6 will examine the cases with $A = 0$.

### *4.1. Discrete cases with $F = 0$ and $E \neq 0$.*

In this subsection, we examine the subcase of (4.1) having $F = 0$ and $E \neq 0$. The analysis of this subcase is similar to the uniqueness proof of Chazy Classes IX–XII among those third-order differential equations that have a $y^4$ term (see Bureau [8] for details). There, the main Diophantine equation, $1/N_1 + 1/N_2 + 1/N_3 = 1/6$, had 141 solutions plus a one-parameter family, but an extra constraint discovered by Chazy allowed 129 of these cases to be eliminated. Here, the corresponding Diophantine equation (4.18) below has 1078 solutions plus a one-parameter family and the corresponding constraint (4.21) below allows 971 cases to be eliminated. The successful handling of this subcase required modest computing power. We did the calculations initially with a 1982-vintage programmable pocket calculator and repeated them on a modern desktop workstation with the computer algebra package REDUCE.

We begin by substituting the trial Laurent series,

$$y = \frac{a_0}{x - x_0} + a_1 + a_2(x - x_0) + \ldots + a_n(x - x_0)^{n-1} + \ldots, \tag{4.10}$$

where $x_0$ is a constant of integration, into (4.1) with $F = 0$. We find that $a_0$ satisfies the cubic equation,

$$E(a_0)^3 - (2C + D)(a_0)^2 + 2(3A + B)a_0 + 24 = 0. \tag{4.11}$$

According to a theorem of Martynov [21], the roots must be distinct. Let the roots be denoted $a_{01}$, $a_{02}$, and $a_{03}$ and let the respective triads of resonance numbers be denoted $\{n_1, n_2, n_3\}$, $\{m_1, m_2, m_3\}$, and $\{p_1, p_2, p_3\}$.

A standard calculation shows that the three sets of resonance numbers are the roots of the three respective cubic equations,

$$\begin{aligned} n^3 &- \{11 + Aa_{01}\}n^2 + \{46 + (7A + B)a_{01} - C(a_{01})^2\}n \\ &\qquad - \{72 + 4(3A + B)a_{01} - (2C + D)(a_{01})^2\} = 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} m^3 &- \{11 + Aa_{02}\}m^2 + \{46 + (7A + B)a_{02} - C(a_{02})^2\}m \\ &\qquad - \{72 + 4(3A + B)a_{02} - (2C + D)(a_{02})^2\} = 0, \end{aligned} \tag{4.13}$$

$$\begin{aligned} p^3 &- \{11 + Aa_{03}\}p^2 + \{46 + (7A + B)a_{03} - C(a_{03})^2\}p \\ &\qquad - \{72 + 4(3A + B)a_{03} - (2C + D)(a_{03})^2\} = 0. \end{aligned} \tag{4.14}$$

Since the symmetric functions of the roots must be integers, we see at once that $Aa_{01}$, $Aa_{02}$, and $Aa_{03}$ must be integers. Define the products,

$$\begin{aligned} N_1 := n_1 n_2 n_3 &= 72 + 4(3A + B)a_{01} - (2C + D)(a_{01})^2 \\ &= 24\,\frac{(a_{02} - a_{01})(a_{03} - a_{01})}{a_{02}a_{03}}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} N_2 := m_1 m_2 m_3 &= 72 + 4(3A + B)a_{02} - (2C + D)(a_{02})^2 \\ &= 24\,\frac{(a_{01} - a_{02})(a_{03} - a_{02})}{a_{01}a_{03}}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} N_3 := p_1 p_2 p_3 &= 72 + 4(3A + B)a_{03} - (2C + D)(a_{03})^2 \\ &= 24\,\frac{(a_{01} - a_{03})(a_{02} - a_{03})}{a_{01}a_{02}}. \end{aligned} \tag{4.17}$$

These products are never 0, $\pm 1$, $\pm p$, or $p^2$, where $p$ is a prime number, because these numbers cannot be factored into admissible resonance numbers.

From the expressions in terms of the $a_{0i}$, we see that $N_1$, $N_2$, and $N_3$ satisfy the Diophantine equation,

$$\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} = \frac{1}{24}\,. \tag{4.18}$$

This equation has the one-parameter family of solutions,

$$N_1 = 24, \qquad N_3 = -N_2\,, \tag{4.19}$$

or a permutation thereof, where $N_2$ is an arbitrary positive integer. The consequences of this infinite family will be explored in the next section. In addition, there are exactly 1078 discrete unordered solutions, which can be subdivided, without loss of generality, as follows:

$$\begin{array}{lll} 24 < N_1 \leq 72, & N_1 \leq N_2 \leq N_3 : & \text{366 solutions,} \\ 0 < N_1 < 48, & N_1 \leq N_2, \quad N_3 < 0 : & \text{476 solutions,} \\ 0 < N_1 < 24, & N_3 \leq N_2 < 0 : & \text{236 solutions.} \end{array}$$

Cases with $N_i = 24$, which belong to (4.19), are not counted. If values of $N_i$ that do not admit admissible factorizations into resonance numbers are excluded, the above counts are lowered to 284, 342, and 171, respectively, and the total is lowered from 1078 to 797.

In the case $A \neq 0$, an additional constraint is available. From (4.15)–(4.17), we see that

$$-\tfrac{1}{6} N_1 N_2 N_3 = \frac{48^2 (a_{01} - a_{02})^2 (a_{01} - a_{03})^2 (a_{02} - a_{03})^2}{(a_{01})^2 (a_{02})^2 (a_{03})^2}\,. \tag{4.20}$$

When $A \neq 0$, the numbers $Aa_{01}$, $Aa_{02}$, and $Aa_{03}$ are nonzero integers. Hence the right-hand side of (4.20) is the square of a rational number. But this forces the left-hand side to be the square of an integer. Hence we have the constraint,

$$N_1 N_2 N_3 = -6q^2, \tag{4.21}$$

where $q$ is a positive integer. Only 107 discrete solutions of the Diophantine equation (4.18) satisfy this additional constraint. Of these, only 70 have $N_i$ with admissible factorizations into resonance numbers. The solutions of (4.18) to be examined have now been reduced to a manageable number.

The 70 unordered triads $\{N_1, N_2, N_3\}$ solving equation (4.18) can be subdivided into two families, each member of which has $N_1 > 0$, $N_2 N_3 < 0$, and $N_1 \le |N_2| \le |N_3|$. The first family, which has 27 members, satisfies the inequalities,

$$5 \le N_1 \le 22, \qquad \frac{24N_1}{N_1 - 24} \le N_2 \le \frac{24N_1}{N_1 - 48}, \tag{4.22}$$

and the second family, which has 43 members, satisfies the inequalities,

$$26 \le N_1 \le 46, \qquad N_1 \le N_2 < \frac{24N_1}{N_1 - 24}. \tag{4.23}$$

The members of the first family are

$$\{10, -16, 240\}, \quad \{12, -21, 168\}, \quad \{14, -30, 280\}, \quad \{14, -32, 672\},$$
$$\{15, -24, 60\}, \quad \{15, -36, 360\}, \quad \{15, -39, 1560\}, \quad \{16, -30, 80\},$$
$$\{16, -40, 240\}, \quad \{16, -46, 1104\}, \quad \{18, -48, 144\}, \quad \{18, -66, 792\},$$
$$\{20, -56, 105\}, \quad \{20, -84, 280\}, \quad \{20, -95, 456\}, \quad \{20, -104, 780\},$$
$$\{20, -111, 1480\}, \quad \{20, -116, 3480\}, \quad \{20, -119, 14280\}, \quad \{21, -120, 420\},$$
$$\{21, -156, 2184\}, \quad \{21, -165, 9240\}, \quad \{22, -192, 704\}, \quad \{22, -232, 1914\},$$
$$\{22, -246, 3608\}, \quad \{22, -256, 8448\}, \quad \{22, -262, 34584\}.$$

The members of the second family are

$$\{26, 240, -1040\}, \quad \{26, 280, -2730\}, \quad \{26, 294, -5096\}, \quad \{26, 304, -11856\},$$
$$\{26, 310, -48360\}, \quad \{27, 108, -216\}, \quad \{27, 168, -756\}, \quad \{27, 189, -1512\},$$
$$\{27, 204, -3672\}, \quad \{27, 213, -15336\}, \quad \{28, 104, -273\}, \quad \{28, 119, -408\},$$
$$\{28, 132, -616\}, \quad \{28, 152, -1596\}, \quad \{28, 159, -2968\}, \quad \{28, 164, -6888\},$$
$$\{30, 96, -480\}, \quad \{30, 114, -2280\}, \quad \{32, 64, -192\}, \quad \{32, 78, -416\},$$
$$\{32, 88, -1056\}, \quad \{32, 94, -4512\}, \quad \{33, 72, -396\}, \quad \{33, 84, -1848\},$$
$$\{33, 87, -7656\}, \quad \{34, 78, -1768\}, \quad \{34, 80, -4080\}, \quad \{35, 40, -84\},$$
$$\{35, 76, -15960\}, \quad \{36, 45, -120\}, \quad \{36, 60, -360\}, \quad \{36, 69, -1656\},$$
$$\{38, 64, -3648\}, \quad \{39, 60, -1560\}, \quad \{40, 44, -165\}, \quad \{40, 51, -340\},$$
$$\{40, 56, -840\}, \quad \{42, 48, -336\}, \quad \{42, 54, -1512\}, \quad \{44, 51, -1496\},$$
$$\{44, 52, -3432\}, \quad \{45, 51, -6120\}, \quad \{46, 50, -13800\}.$$

There are several ways to proceed at this point. We need to find admissible sets of resonance numbers $\{n_1, n_2, n_3\}$, $\{m_1, m_2, m_3\}$, and $\{p_1, p_2, p_3\}$ such that

$$N_1 = n_1 n_2 n_3 , \qquad N_2 = m_1 m_2 m_3 , \qquad N_3 = p_1 p_2 p_3 . \tag{4.24}$$

The most efficient method involves listing all the factorizations of $N_1$ only, which is the smallest of the $N_i$, into admissible resonance numbers for each of the 70 triads $\{N_1, N_2, N_3\}$. There are 293 such factorizations, including repetitions where the same $N_1$ appears in several triads. Thus we may treat the $n_i$ as known quantities, since we have a finite list of candidates. At this stage we do not know the $m_i$ individually but we do know their product $N_2$. Form the symmetric functions,

$$Q_1 = n_1 + n_2 + n_3 , \qquad R_1 = n_1 n_2 + n_1 n_3 + n_2 n_3 , \tag{4.25}$$

$$Q_2 = m_1 + m_2 + m_3 , \qquad R_2 = m_1 m_2 + m_1 m_3 + m_2 m_3 . \tag{4.26}$$

Of course, similar symmetric functions can be formed from the $p_i$ as well. The differential equation is, in general, overdetermined when five or more resonance numbers are assigned.

Use the scaling freedom in the differential equation to make $a_{01} = 1$. This forces $A$ to be an integer. When $A \neq 0$, which is the case we are considering here, it also forces $a_{02}$ and $a_{03}$ to be rational numbers. According to (4.11), we find that

$$E = 2C + D - 2(3A + B) - 24 \tag{4.27}$$

and that $a_{02}$ and $a_{03}$ are the roots of the quadratic equation,

$$(2C + D)(a_0)^2 - 2(3A + B)a_0(a_0 + 1) - 24\big((a_0)^2 + a_0 + 1\big) = 0. \tag{4.28}$$

Equation (4.12) gives

$$A = Q_1 - 11, \tag{4.29}$$

$$C = B + 7Q_1 - R_1 - 31, \tag{4.30}$$

$$D = 2B - 2Q_1 + 2R_1 - N_1 + 2, \tag{4.31}$$

$$E = 2B + 6Q_1 - N_1 - 18. \tag{4.32}$$

The coefficient $B$ remains undetermined when just the first set of resonance numbers $\{n_1, n_2, n_3\}$ is assigned. Similarly, from (4.13), we find

$$A a_{02} = Q_2 - 11, \tag{4.33}$$

$$C(a_{02})^2 = B a_{02} + 7Q_2 - R_2 - 31, \tag{4.34}$$

$$D(a_{02})^2 = 2B a_{02} - 2Q_2 + 2R_2 - N_2 + 2, \tag{4.35}$$

$$E(a_{02})^3 = 2B a_{02} + 6Q_2 - N_2 - 18. \tag{4.36}$$

The last equation can be replaced by (4.28) with $a_0 = a_{02}$.

If we eliminate $A$, $B$, $C$, $D$, $E$, $Q_2$, and $R_2$ from equations (4.27)–(4.36), we obtain the quadratic equation for $a_{02}$,

$$(N_1 - 24)(a_{02})^2 + 48a_{02} + N_2 - 24 \;=\; 0, \tag{4.37}$$

in which the coefficients are known. Since $a_{02}$ must be rational, it follows that $24(N_1 + N_2) - N_1N_2$ is a perfect square. This quantity can also be written $-24N_1N_2/N_3$. Hence, by symmetry, all the quantities,

$$-24N_1N_2/N_3\,, \qquad -24N_1N_3/N_2\,, \qquad -24N_2N_3/N_1\,, \tag{4.38}$$

are squares of integers. This appears to place stronger constraints on the $N_i$ than (4.21). However, it does not eliminate any more cases.

Hence $a_{02}$ is one of the roots of (4.37) and both roots are already rational. We may now consider $a_{02}$ to be known. Note that $N_i \neq 24$ implies that $a_{02}$ never takes the inadmissible values 0 or $a_{01}$. Finally, (4.28) with $a_0 = a_{02}$ gives

$$B \;=\; \frac{3(3a_{02} - 4)}{a_{02}} \;+\; \frac{\frac{1}{2}N_1a_{02}}{a_{02} - 1} \;-\; 3Q_1\,. \tag{4.39}$$

Thereafter, $a_{03}$ can be obtained by dividing the left-hand side of (4.28) by $a_0 - a_{02}$. We have now formed $293 \times 2 = 586$ differential equations that satisfy a partial set of necessary conditions for integer resonances.

The remaining conditions are that all the roots of the cubic equations (4.13) and (4.14) be integers. The roots of (4.13) are integers in just 8 of the 586 cases. The roots of both (4.13) and (4.14) are integers in just a single case having $N_1 = 15$, $N_2 = -24$, and $N_3 = 60$.

The unique surviving case is the reduced form of equation F-XV:

$$\text{F-XV}: \qquad y^{(\mathrm{iv})} \;=\; -\,2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y'. \tag{4.40}$$

It has the following resonance data:

$$a_{01} = 1: \;\; \{1, 3, 5\}, \qquad a_{02} = 4: \;\; \{1, -4, 6\}, \qquad a_{03} = 6: \;\; \{-2, -5, 6\}. \tag{4.41}$$

This equation has the Painlevé property. The full version of F-XV and its general solution is given in Section 7.

### *4.2. Discrete cases with $F \neq 0$.*

Equation (4.1) with $F \neq 0$ can be analysed in a similar fashion to the preceding subsection. However, the Diophantine equations now have a very large number of solutions. The principal Diophantine equation (4.51) below has 202511 discrete solutions and several infinite families. The calculations in this section were done on a Digital Alpha Station with the computer-algebra package REDUCE.

On substituting the trial Laurent series (4.10) into (4.1), we find that the leading coefficient $a_0$ satisfies the quartic equation,

$$F(a_0)^4 - E(a_0)^3 + (2C + D)(a_0)^2 - 2(3A + B)a_0 - 24 = 0. \qquad (4.42)$$

Let the roots be denoted $a_{01}$, $a_{02}$, $a_{03}$, and $a_{04}$ and let the corresponding triads of resonance numbers be denoted $\{n_1, n_2, n_3\}$, $\{m_1, m_2, m_3\}$, $\{p_1, p_2, p_3\}$, and $\{q_1, q_2, q_3\}$. A routine calculation shows that the resonance numbers are the roots of the respective cubic equations,

$$\begin{aligned} n^3 &- \{11 + Aa_{01}\}n^2 + \{46 + (7A + B)a_{01} - C(a_{01})^2\}n \\ &- \{96 + 6(3A + B)a_{01} - 2(2C + D)(a_{01})^2 + E(a_{01})^3\} = 0, \qquad (4.43) \\ m^3 &- \{11 + Aa_{02}\}m^2 + \{46 + (7A + B)a_{02} - C(a_{02})^2\}m \\ &- \{96 + 6(3A + B)a_{02} - 2(2C + D)(a_{02})^2 + E(a_{02})^3\} = 0, \qquad (4.44) \\ p^3 &- \{11 + Aa_{02}\}p^2 + \{46 + (7A + B)a_{03} - C(a_{03})^2\}p \\ &- \{96 + 6(3A + B)a_{03} - 2(2C + D)(a_{03})^2 + E(a_{03})^3\} = 0, \qquad (4.45) \\ q^3 &- \{11 + Aa_{02}\}q^2 + \{46 + (7A + B)a_{04} - C(a_{04})^2\}q \\ &- \{96 + 6(3A + B)a_{04} - 2(2C + D)(a_{04})^2 + E(a_{04})^3\} = 0. \qquad (4.46) \end{aligned}$$

These equations show that $Aa_{0i}$, $i = 1, 2, 3, 4$, must be integers. Also, if $A \neq 0$, the ratios $a_{0i}/a_{01}$ are rational. Define the products,

$$\begin{aligned} N_1 &= n_1 n_2 n_3 = 96 + 6(3A + B)a_{01} - 2(2C + D)(a_{01})^2 + E(a_{01})^3 \\ &= 24\,\frac{(a_{02} - a_{01})(a_{03} - a_{01})(a_{04} - a_{01})}{a_{02}a_{03}a_{04}}, \qquad (4.47) \\ N_2 &= m_1 m_2 m_3 = 96 + 6(3A + B)a_{02} - 2(2C + D)(a_{02})^2 + E(a_{02})^3 \\ &= 24\,\frac{(a_{01} - a_{02})(a_{03} - a_{02})(a_{04} - a_{02})}{a_{01}a_{03}a_{04}}, \qquad (4.48) \\ N_3 &= p_1 p_2 p_3 = 96 + 6(3A + B)a_{03} - 2(2C + D)(a_{03})^2 + E(a_{03})^3 \\ &= 24\,\frac{(a_{01} - a_{03})(a_{02} - a_{03})(a_{04} - a_{03})}{a_{01}a_{02}a_{04}}, \qquad (4.49) \end{aligned}$$

$$\begin{aligned} N_4 \;=\; q_1q_2q_3 \;&=\; 96 + 6(3A+B)a_{04} - 2(2C+D)(a_{04})^2 + E(a_{04})^3 \\ &=\; 24\,\frac{(a_{01}-a_{04})(a_{02}-a_{04})(a_{03}-a_{04})}{a_{01}a_{02}a_{03}}\,. \end{aligned} \tag{4.50}$$

As before, these products cannot take the values 0, $\pm 1$, $\pm p$, or $p^2$, where $p$ is a prime.

From the expressions in terms of the $a_{0i}$, we see that the $N_i$ satisfy the Diophantine equation,

$$\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4} \;=\; \frac{1}{24}\,. \tag{4.51}$$

This equation, which was first found by Chazy [6], has exactly 202511 discrete unordered solutions where $N_i \neq 24$ and no pair $\{N_i, N_j\}$ satisfies $1/N_i + 1/N_j = 1/24$. There are also two types of infinite families of solutions. The first type takes the form,

$$\frac{1}{N_1} + \frac{1}{N_2} = \frac{1}{24}\,, \qquad N_4 \;=\; -N_3\,, \tag{4.52}$$

or a permutation thereof, where $N_3$ is an arbitrary positive integer. There are 21 families of this type, corresponding to the 21 pairs $\{N_1, N_2\}$. Both the discrete solutions and these infinite families will be examined in this subsection. The second type of infinite family is given by

$$N_1 \;=\; 24, \qquad \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4} \;=\; 0, \tag{4.53}$$

or a permutation thereof. The integer $N_2$ can be chosen arbitrarily. Thence there is a finite positive number of pairs $\{N_3, N_4\}$ corresponding to each $N_2$, and that number is large when $N_2$ has many divisors. This case, which cannot be handled by traditional methods, is postponed till the next section.

The discrete solutions of (4.51) can be subdivided in the following manner:

| | |
|---|---|
| $24 < N_1 \le 96,\quad N_1 \le N_2 \le N_3 \le N_4:$ | 38904 solutions, |
| $0 < N_1 < 72,\quad N_1 \le N_2 \le N_3,\quad N_4 < 0:$ | 77738 solutions, |
| $0 < N_1 < 48,\quad N_1 \le N_2,\quad N_4 \le N_3 < 0:$ | 63940 solutions, |
| $0 < N_1 < 24,\quad N_4 \le N_3 \le N_2 < 0:$ | 21929 solutions. |

(These counts have been carried out to satisfy mathematical curiosity; they are not essential to our work.) We have not found a simplifying constraint analogous to (4.21). However, it is possible to complete the analysis by examining just two of the $N_i$, these being, in general, the two smallest in absolute value. The aim is to construct a manageable list of differential equations that contains all the relevant Painlevé equations by a selective use of part of the information contained in equations (4.42)–(4.51). When that is done, each differential equation can be tested directly for integer resonances and the spurious ones discarded.

First, we let $N_1$ run over the 66 integer values in the range $1 < N_1 \leq 96$, excluding $N_1 = 24$, that can be factorized legally into three resonance numbers. By reordering the other $N_i$ if necessary, we can ensure that $N_2$ is contained within the following bounds:

$$6 \leq N_1 \leq 22: \qquad \frac{72N_1}{N_1 - 24} \leq N_2 \leq \frac{24N_1}{N_1 - 72}, \tag{4.54}$$

$$26 \leq N_1 \leq 96: \qquad N_1 \leq N_2 \leq \frac{72N_1}{N_1 - 24}. \tag{4.55}$$

We emphasize that these inequalities capture not just the discrete solutions but also the 21 infinite families (4.52). They are also not the best-possible bounds on $N_2$. We could sharpen these bounds at the cost of splitting the problem into additional subcases. There are 11 and 55 admissible values of $N_1$ in the respective ranges. We only need to know the bounds on $N_2$, not the actual values that $N_2$ can take according to the Diophantine equation.

The first step is to prepare a table of triads $\{n_1, n_2, n_3\}$ where

$$n_1 n_2 n_3 = N_1, \qquad n_1 + n_2 + n_3 \neq 11.$$

The latter condition is imposed because we are setting aside the case $A = 0$ until Section 6. Exactly 409 triads occur in range. This will allow us to construct 409 differential equations, each containing two undetermined parameters after we use scaling freedom to set $a_{01} = 1$. According to (4.42), the latter normalization yields the constraint,

$$F = 6A + 2B - 2C - D + E + 24, \tag{4.56}$$

and implies that $a_{02}$, $a_{03}$, and $a_{04}$ are the roots of the cubic equation,

$$F(a_0)^3 + (F - E)(a_0)^2 + 2(3A + B + 12)a_0 + 24 = 0. \tag{4.57}$$

Let $Q_1$ and $R_1$ be defined by (4.25). We treat $Q_1$, $R_1$, and $N_1$ as known quantities. Then the cubic equation (4.43) for the $n_i$ gives the coefficients of the differential equation (4.1) in terms of $B$ and $D$, which remain undetermined at this stage. The results are

$$A = Q_1 - 11, \tag{4.58}$$

$$C = B + 7Q_1 - R_1 - 31, \tag{4.59}$$

$$E = 2D - 2B + 10Q_1 - 4R_1 + N_1 - 22, \tag{4.60}$$

$$F = D - 2B + 2Q_1 - 2R_1 + N_1 - 2. \tag{4.61}$$

Similarly, from the cubic equation (4.44) for the second set of resonances, we may deduce the corresponding relations,

$$Aa_{02} = Q_2 - 11, \tag{4.62}$$

$$C(a_{02})^2 = Ba_{02} + 7Q_2 - R_2 - 31, \tag{4.63}$$

$$E(a_{02})^3 = 2D(a_{02})^2 - 2Ba_{02} + 10Q_2 - 4R_2 + N_2 - 22, \tag{4.64}$$

$$F(a_{02})^4 = D(a_{02})^2 - 2Ba_{02} + 2Q_2 - 2R_2 + N_2 - 2, \tag{4.65}$$

where $N_2 = m_1 m_2 m_3$ and $Q_2$ and $R_2$ are defined by (4.26). These equations will be useful also in the next two sections.

For each $N_1$, let $M$ denote the upper bound on $|N_2|$ according to (4.54) and (4.55). Then $|m_1 m_2 m_3| \le M$. Order the $m_i$ so that $|m_1| \le |m_2| \le |m_3|$. Then

$$|m_1| \le M^{1/3},$$

$$|m_1| \le |m_2| \le \sqrt{M/|m_1|},$$

$$|m_2| \le |m_3| \le M/|m_1 m_2|.$$

Additional restrictions are that

$$m_i \ne 0, \qquad m_i \ne -1, \qquad m_i \ne m_{i-1},$$

$$m_1 + m_2 + m_3 \ne 11, \qquad m_1 + m_2 + m_3 \ne Q_1,$$

and, without loss of generality, $m_i \ne |m_{i-1}|$. In this fashion, corresponding to each of the 409 triads $\{n_1, n_2, n_3\}$, a finite list of triads $\{m_1, m_2, m_3\}$ can be generated.

Next, from (4.58)–(4.64), we deduce that

$$a_{02} = \frac{Q_2 - 11}{Q_1 - 11}, \tag{4.66}$$

$$B = \frac{(R_1 - 7Q_1 + 31)(a_{02})^2 - (R_2 - 7Q_2 + 31)}{a_{02}(a_{02} - 1)}, \tag{4.67}$$

$$\begin{aligned} D = {} & B\,\frac{a_{02}+1}{a_{02}} \\ & + \frac{(4R_1 - 10Q_1 - N_1 + 22)(a_{02})^3 - (4R_2 - 10Q_2 - N_2 + 22)}{2(a_{02})^2(a_{02} - 1)}. \end{aligned} \tag{4.68}$$

We now have a finite list of sets $\{A, B, C, D, E, F\}$ of differential equation coefficients together with corresponding values for $a_{02}$ and the triads of resonance numbers $\{n_1, n_2, n_3\}$ and $\{m_1, m_2, m_3\}$. Cases with $Q_2 = Q_1$ have been discarded because $a_{02}$ must not equal $a_{01}$. Also cases with $Q_2 = 11$ can only occur when $Q_1 = 11$, these cases being set aside till Section 6.

The list of differential equations so far has length 280236. This precise value is not significant because we have not yet used all of the applicable constraints and because the list includes a substantial number of repetitions and also pairs of differential equations equivalent under a rescaling on account of the bounds on $N_2$ being not best possible. It is not worthwhile to attempt to remove redundant equations at this stage.

First, the main constraint that we have not yet used is (4.65), which is equivalent to the requirement that $a_{02}$ be one of the roots of equation (4.57). This constraint is severe and shortens the list to length 478. This is now a manageable size. The list consists of differential equations, with some repetition, having two sets of integer resonances. It is not the complete list with that property, but it is guaranteed to contain all cases having four sets of integer resonances, excluding those cases that have been set aside till the next two sections.

Secondly, 11 of these equations have $F = 0$, which is a case already treated in the previous subsection and section. Hence we discard these, leaving a list of differential equations of length 467. Thirdly, because $A \neq 0$ here, all of the roots of the cubic equation (4.57) must be rational and distinct, these roots being $a_{02}$, $a_{03}$, and $a_{04}$. This restriction further shortens the list to length 83. It is now a simple matter to identify and remove duplicate equations, leaving a list with 74 equations. Fourthly, we may calculate $N_3$ and $N_4$ from (4.49) and (4.50). The condition that $N_3$ and $N_4$ both be integers eliminates 52 equations, leaving just 22 equations. Of these, 19 correspond to discrete solutions of (4.51) and 3 correspond to particular members of the family (4.52).

The coefficients $\{A, B, C, D, E, F\}$ of the 22 surviving equations are

$$\big\{-5,-10,-10,-15,-10,-1\big\}, \qquad \big\{-3,-10,-2,-9,2,1\big\},$$
$$\big\{-3,-8,-2,-3,4,1\big\}, \qquad \big\{-2,-10,2,-6,8,2\big\},$$
$$\big\{-2,-10,-1,-9/2,2,1/2\big\}, \qquad \big\{-13,30,-6,33,18,3\big\},$$
$$\big\{-1,-6,6,9,18,3\big\}, \qquad \big\{-1,-10,2,-1,6,1\big\},$$
$$\big\{-1,-8,2,3,4,-1\big\}, \qquad \big\{2,-17,11,-15,4,-1\big\},$$
$$\big\{-1,-5,5,12,16,2\big\}, \qquad \big\{-1,-52/5,-2/5,-3/5,8/5,1/5\big\},$$
$$\big\{3,-17,7,-9/2,1,-1/2\big\}, \qquad \big\{-1,-5,3,7/2,1,-1/2\big\},$$
$$\big\{-11,30,18,-9,6,-3\big\}, \qquad \big\{-11,26,14,-15,2,-1\big\},$$
$$\big\{-1,-4,4,5,2,-1\big\}, \qquad \big\{10,-30,30,-30,0,-6\big\},$$
$$\big\{5,-20,10,-5,0,-1\big\}, \qquad \big\{-24,68,-80,432,512,256\big\},$$
$$\big\{5,-5,0,20,-20,4\big\}, \qquad \big\{10,-20,-20,60,-20,4\big\}.$$

The differential equations of the form (4.1) with these coefficients are the only equations having the following properties: $A \neq 0$, $F \neq 0$, the $N_i$ are integers, $N_i \neq 24$, $N_1$ and $N_2$ lie in the ranges (4.54) or (4.55), and the resonance numbers $\{n_1, n_2, n_3\}$ and $\{m_1, m_2, m_3\}$, whose products are $N_1$ and $N_2$, respectively, are integers obeying the usual rules.

It is now a straightforward matter to test the cubic equations (4.45) and (4.46) for integer roots. These roots form the third and fourth sets of resonance numbers, $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$. We find that both sets contain non-integer resonances or repeated resonances in five cases and that one set contains non-integer resonances in 15 cases. Only two cases have all integer resonances. These are the first and tenth members of the above list.

The first case corresponds to the discrete solution,

$$\big\{N_1, N_2, N_3, N_4\big\} = \big\{6,-4,6,-24\big\}, \tag{4.69}$$

of the Diophantine equation (4.51). The differential equation is the reduced form of equation F-XVI and is given by

$$\text{F-XVI}: \quad y^{(\mathrm{iv})} = -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5. \tag{4.70}$$

This equation has the following resonance data:

$$a_{01} = 1: \quad \{1, 2, 3\}, \qquad a_{02} = 2: \quad \{1, 2, -2\},$$
$$a_{03} = 3: \quad \{1, -2, -3\}, \qquad a_{04} = 4: \quad \{-2, -3, -4\}.$$

The reduced equation F-XVI was discovered in the context of Painlevé classification by both Martynov [21] and Exton [25], and the latter author was aware that this equation is trivially linearizable by the substitution $y = w'/w$. We attach recessive terms to this equation in Section 7.

The second case corresponds to the solution,

$$\{N_1, N_2, N_3, N_4\} = \{20, -60, 60, -120\}, \tag{4.71}$$

of the Diophantine equation (4.51). This is a particular member of the infinite family (4.52) because $N_3 = -N_2$. The differential equation is new and is given by

$$y^{(\mathrm{iv})} = 2yy''' - 17y'y'' + 11y^2y'' - 15y(y')^2 + 4y^3y' - y^5. \tag{4.72}$$

This equation has the following integer resonance data:

$$a_{01} = 1: \quad \{1, 2, 10\}, \qquad a_{02} = 2: \quad \{2, -2, 15\},$$
$$a_{03} = -3: \quad \{-2, -3, 10\}, \qquad a_{04} = -4: \quad \{1, -10, 12\}.$$

However, like equation (3.49) above, it fails the freedom-from-logarithms test at the last resonance $n = 10$ when $a_0 = a_{01}$. It is therefore not a Painlevé-type equation. Nevertheless, it may be worthy of further study because of its close connection to the Chazy-XII equation, as mentioned at the beginning of this section.

## 5. The infinite families having $N_1 = 24$

In this section, we construct all Painlevé-type differential equations of the form (4.1), in which at least one of the coefficients $A, \ldots, F$ is nonzero, such that one set of resonance numbers $\{n_1, n_2, n_3\}$ has the property,

$$N_1 := n_1 n_2 n_3 = 24. \tag{5.1}$$

If the differential equation (4.1) takes the form (3.1) or (3.2), then there is only one set of resonance numbers and that set necessarily obeys (5.1). These cases have already been fully analysed in Section 3, but will appear again here under more general hypotheses. On the other hand, equation (3.3), which was also fully analysed in Section 3, has two sets of resonance numbers, but it is impossible for either of them to satisfy (5.1) because of the governing Diophantine equation (3.39).

If $F = 0$ and $E \neq 0$, then there are three sets of resonance numbers whose products $N_i$ satisfy the Diophantine equation (4.18). Similarly, if $F \neq 0$, then there are four sets of resonance numbers whose products satisfy the Diophantine equation (4.51). Cases having $N_1 = 24$ belong to the infinite families (4.19) and (4.53), respectively, that were set aside in Section 4 for analysis in this section. The other infinite family (4.52) was fully analysed in the last section along with the discrete cases.

The infinite families (4.19) and (4.53), especially the latter, would be very difficult if not impossible to analyse in full by traditional methods based on obtaining conditions for integer resonance numbers. The methods of Subsection 4.2 could be employed if we placed an arbitrary search limit on $N_2$, say $|N_2| \leq 10000$, or search limits on the resonance numbers $m_i$ that divide $N_2$. If we did that, we would certainly find all the relevant Painlevé equations but we would not have a proof of their uniqueness. Some subcases of (4.19) are quite easy by traditional methods. Others run into the problem of finding rational points on elliptic curves and draw upon elementary and algebraic number theory. It is easy to get sidetracked into these beautiful branches of mathematics, but they are the wrong methods to apply here. They do not generalize to higher-order problems.

The methods we present in this section and the next give the complete solutions of the respective problems with surprisingly little effort. In fact, the computer searches in Subsection 4.2 could be shortened significantly by taking out the cases where $N_1$ is near 24, especially $N_1 = 22$ and $N_1 = 26$, and applying the methods of the present section. Such a move is highly recommended at the corresponding stage of the fifth-order Painlevé classification problem, where the critical value of $N_1$ is 120. The key that makes this possible is the CFP test of Conte, Fordy, and Pickering [9, 10]. Thus the CFP test should be regarded as an indispensible tool for those researchers in the business of Painlevé classification.

The number $N_1 = 24$ has exactly 11 legal factorizations into resonance numbers $\{n_1, n_2, n_3\}$. These factorizations are

$$\{1, 2, 12\}, \qquad \{1, 3, 8\}, \qquad \{1, 4, 6\}, \qquad \{2, 3, 4\}, \tag{5.2}$$

$$\{1, -2, -12\}, \qquad \{1, -3, -8\}, \qquad \{1, -4, -6\}, \tag{5.3}$$

$$\{2, -2, -6\}, \qquad \{2, -3, -4\}, \qquad \{3, -2, -4\}, \qquad \{4, -2, -3\}. \tag{5.4}$$

Now when one set of resonance numbers is known, the differential equation itself is known up to two undetermined coefficients. If $B$ and $D$ are chosen as the undetermined coefficients, the remaining coefficients are given by (4.58)–(4.61), where $Q_1$

and $R_1$ are defined by (4.25).

The 11 corresponding differential equations under investigation here are

$$y^{(\mathrm{iv})} = 4yy''' + By'y'' + (B+36)y^2y'' + Dy(y')^2 + 2(D-B)y^3y' + (D-2B-24)y^5, \tag{5.5}$$

$$y^{(\mathrm{iv})} = yy''' + By'y'' + (B+18)y^2y'' + Dy(y')^2 + 2(D-B-9)y^3y' + (D-2B-24)y^5, \tag{5.6}$$

$$y^{(\mathrm{iv})} = By'y'' + (B+12)y^2y'' + Dy(y')^2 + 2(D-B-12)y^3y' + (D-2B-24)y^5, \tag{5.7}$$

$$y^{(\mathrm{iv})} = -2yy''' + By'y'' + (B+6)y^2y'' + Dy(y')^2 + 2(D-B-6)y^3y' + (D-2B-12)y^5, \tag{5.8}$$

$$y^{(\mathrm{iv})} = -24yy''' + By'y'' + (B-132)y^2y'' + Dy(y')^2 + 2(D-B-84)y^3y' + (D-2B-24)y^5, \tag{5.9}$$

$$y^{(\mathrm{iv})} = -21yy''' + By'y'' + (B-114)y^2y'' + Dy(y')^2 + 2(D-B-75)y^3y' + (D-2B-24)y^5, \tag{5.10}$$

$$y^{(\mathrm{iv})} = -20yy''' + By'y'' + (B-108)y^2y'' + Dy(y')^2 + 2(D-B-72)y^3y' + (D-2B-24)y^5, \tag{5.11}$$

$$y^{(\mathrm{iv})} = -17yy''' + By'y'' + (B-69)y^2y'' + Dy(y')^2 + 2(D-B-21)y^3y' + (D-2B+18)y^5, \tag{5.12}$$

$$y^{(\mathrm{iv})} = -16yy''' + By'y'' + (B-64)y^2y'' + Dy(y')^2 + 2(D-B-20)y^3y' + (D-2B+16)y^5, \tag{5.13}$$

$$y^{(\mathrm{iv})} = -14yy''' + By'y'' + (B-42)y^2y'' + Dy(y')^2 + 2(D-B+6)y^3y' + (D-2B+36)y^5, \tag{5.14}$$

$$y^{(\mathrm{iv})} = -12yy''' + By'y'' + (B-24)y^2y'' + Dy(y')^2 + 2(D-B+24)y^3y' + (D-2B+48)y^5. \tag{5.15}$$

We will now study each of these differential equations in turn. The main idea is to determine additional constraints on $B$ and $D$ that are implied by freedom-from-logarithms conditions deduced from the limited resonance data available so far. In all cases, we have used scaling freedom to set $a_{01} = 1$. With the exception of equation (5.7), which has $A = 0$, all the $a_{0i}$, as well as $B$ and $D$, must be rational numbers, the $a_{0i}$ being the roots of the (at most) quartic equation (4.42). (More precisely, the products $Aa_{0i}$ must be integers, whatever the scaling is.) This observation will allow some calculations to be terminated quickly. However, in the case of (5.7), we must be alert to the possibility that some of the $a_{0i}$ or $B$ or $D$ might be irrational (including nonreal).

The first four equations (5.5)–(5.8) have positive resonances, where we have placed the $n_i$ in ascending order. These can be analysed by well-established methods, albeit in a novel context. Substitute the trial Laurent series (4.10), with $a_0 = 1$, into these four equations. The coefficients $a_n$ satisfy a recurrence relation of the form,

$$(n+1)(n-n_1)(n-n_2)(n-n_3)a_n \;=\; \Phi_n(a_0, a_1, \ldots, a_{n-1}), \tag{5.16}$$

for $n \geq 1$. The first nonzero coefficient after $a_0$ is $a_{n_1}$, which is an arbitrary constant of integration along with the position $x_0$ of the pole and the later coefficients $a_{n_2}$ and $a_{n_3}$. The conditions for freedom from logarithms are

$$\Phi_{n_2} \;=\; 0, \qquad \Phi_{n_3} \;=\; 0, \tag{5.17}$$

after the arguments of $\Phi_{n_2}$ are expressed in terms of $a_{n_1}$ only and the arguments of $\Phi_{n_3}$ are expressed in terms of $a_{n_1}$ and $a_{n_2}$ only. Note that the $\Phi_n$ will contain no nonzero terms for $1 \leq n \leq 2n_1 - 1$.

For equation (5.5), having resonances $\{1, 2, 12\}$, the conditions for the absence of logarithms are

$$\begin{aligned}
\Phi_2 \;&=\; 4(D - 3B - 42)(a_1)^2 \;=\; 0, \\
\Phi_{12} \;&=\; \{\ldots\}(a_1)^{12} + \{\ldots\}(a_1)^{10}a_2 + \ldots + \{\ldots\}(a_2)^6 \;=\; 0,
\end{aligned}$$

where the coefficients in the second case are polynomials in $B$ and $D$. The first condition gives the immediate constraint,

$$D \;=\; 3(B + 14).$$

Then the only value of $B$ that makes $\Phi_{12}$ vanish is

$$B \;=\; -10.$$

To get this value, it is sufficient to look at one term, say the $(a_1)^2(a_2)^5$ term, whose coefficient is a polynomial in $B$ of degree 6. Since $B$ must be rational and this polynomial has just one rational root $B = -10$, we need look no further. Alternatively, if one does not have access to computer algebra, one could shorten the calculation by setting $a_1 = 0$ after calculating $\Phi_2$. Then one would find that

$$\begin{aligned}\Phi_{12} \;=\; & \tfrac{3}{81920}\,(B+10)(B+18) \\ & \quad \times (703B^3 + 47054B^2 + 1054420B + 7914600)(a_2)^6,\end{aligned}$$

which vanishes for two rational values of $B$.

The analysis in the previous paragraph has uniquely determined the coefficients in equation (5.5). They are given by

$$\{A, B, C, D, E, F\} \;=\; \{4, -10, 26, 12, 44, 8\}. \tag{5.18}$$

If the shortcut in the previous paragraph is taken, the loss of generality requires us to consider a second possibility, namely,

$$\{A, B, C, D, E, F\} \;=\; \{4, -18, 18, -12, 12, 0\}. \tag{5.19}$$

In the case of (5.18), there are four values of $a_0$ satisfying (4.42), namely,

$$a_{01} \;=\; 1, \qquad a_{02} \;=\; 2, \qquad a_{03} \;=\; 3, \qquad a_{04} \;=\; -\tfrac{1}{2}.$$

The first three give rise to integer resonance numbers. But the cubic equation (4.46) for the fourth set of resonance numbers admits two irrational roots $2 \pm i\sqrt{13/2}$. Similarly, the case (5.19) has three sets of resonance numbers, of which two contain irrational members. Either way, we conclude that equation (5.5) can never have the Painlevé property.

The next three differential equations are handled similarly, except that (5.7) may have irrational coefficients. For equation (5.6), having resonances $\{1, 3, 8\}$, the condition $\Phi_3 = 0$ is quadratic in $D$ and yields two possibilities,

$$D \;=\; 3(B+11), \qquad D \;=\; \tfrac{1}{2}(5B+62).$$

Thence the condition $\Phi_8 = 0$ uniquely determines $B$ in each case, the respective values being $B = -9$ and $B = -10$. We arrive at two differential equations whose coefficients $\{A, B, C, D, E, F\}$ are given by

$$\{1, -9, 9, 6, 12, 0\}, \qquad \{1, -10, 8, 6, 14, 2\}. \tag{5.20}$$

In both cases, the value $a_0 = -1$ gives rise to irrational resonance numbers, and we conclude that equation (5.6) never has the Painlevé property.

For equation (5.7), having resonances $\{1, 4, 6\}$, the condition $\Phi_4 = 0$ is cubic in $D$ and yields three possibilities,

$$D \;=\; 2(B+12), \qquad D \;=\; 3(B+10), \qquad D \;=\; \tfrac{1}{19}(49B+578).$$

Thence the condition $\Phi_6 = 0$ gives two values of $B$ in each case, all of which are rational. We arrive at six differential equations whose coefficients are given by

$$\big\{0, -12, 0, 0, 0, 0\big\}, \qquad \big\{0, -10, 2, 4, 4, 0\big\}, \tag{5.21}$$

$$\big\{0, -8, 4, 6, 4, -2\big\}, \qquad \big\{0, -9, 3, 3, 0, -3\big\}, \tag{5.22}$$

$$\big\{0, -35/3, 1/3, 1/3, 0, -1/3\big\}, \qquad \big\{0, -52/5, 8/5, 18/5, 4, 2/5\big\}. \tag{5.23}$$

The first member of (5.21) yields the differential equation,

$$\text{F-VII}: \qquad y^{(\text{iv})} \;=\; -\,12y'y'', \tag{5.24}$$

which has already appeared in Section 3 and is known to have the Painlevé property. It is the reduced form of equation F-VII. The other five cases all admit irrational resonances and so none of them have the Painlevé property.

For equation (5.8), having resonances $\{2, 3, 4\}$, the condition $\Phi_3 = 0$ is trivially satisfied and the condition $\Phi_4 = 0$ gives the single constraint,

$$D \;=\; 2(B + 6).$$

We arrive at a one-parameter family of differential equations belonging to the family (4.1) having $F = 0$. The special case $B = -6$ is aleady known to be a Painlevé-type equation. It is the reduced form of F-VIII given by

$$\text{F-VIII}: \qquad y^{(\text{iv})} \;=\; -\,2yy''' - 6y'y''. \tag{5.25}$$

If $B \neq -6$, there are three sets of resonance numbers, all of which are integers except for two that are algebraic functions of $B$. By setting,

$$B \;=\; -\,6\,\frac{N^2 - 9}{N^2 - 1}\,, \qquad D \;=\; \frac{96}{N^2 - 1}\,,$$

the remaining resonance numbers become $N$ and $-N$, where $N$ is a positive integer different from 1, 4, or 6. The differential equation now takes the form,

$$\text{F-X/XI}: \qquad y^{(\text{iv})} \;=\; -\,2yy''' - 6y'y'' + \frac{48}{N^2 - 1}\big(y' + y^2\big)\big(y'' + 2yy'\big). \tag{5.26}$$

It is the reduced equation for both F-X and F-XI, these being separated because the recessive terms behave in two distinct ways. This matter is taken up in Section 7. The resonance data for this equation are

$$a_{01} = 1: \;\; \{2, 3, 4\}, \qquad a_{02} = \tfrac{1}{2}(1 - N): \;\; \{4, 6, N\},$$

$$a_{03} = \tfrac{1}{2}(1 + N): \;\; \{4, 6, -N\}.$$

Equation (5.26) is integrable in terms of Lamé functions, but does not necessarily have the Painlevé property, for unrestricted $N \neq \pm 1$. It has the Painlevé property for $N = 2$ and all odd $N \geq 3$. A first integral of (5.26) is

$$y''' \;=\; -\,2yy'' - 2(y')^2 + \frac{24}{N^2-1}\left(y' + y^2\right)^2 + K, \tag{5.27}$$

where $K$ is a constant of integration. Without the additive constant, this is the reduced Chazy-XI equation. Two further integrations give the Riccati equation,

$$y' + y^2 \;=\; \tfrac{1}{4}(N^2 - 1)\wp(x - x_0, g_2, g_3), \tag{5.28}$$

where $\wp$ is the Weierstrass elliptic function, $x_0$ and $g_3$ are two more integration constants, and $g_2 = -8K/(N^2 - 1)$. The integration of the Riccati equation may introduce logarithmic branch points at the poles of the elliptic function.

A detailed study of the recessive terms admitted by Chazy-XI when the strict Painlevé property is enforced is given in [19], the most interesting cases being $N = 2$, 3, 4, 5, 7, and 11. In Section 7, we will see that equations F-X and F-XI also have analogous interesting cases. It is known that Chazy-XI can admit an additive constant for $N = 2$ and for all odd $N \geq 3$ and that it does not have the Painlevé property when $N$ is a multiple of 6. For other even values of $N$, the claim that Chazy-XI can never take an additive constant is a consequence of a conjecture regarding the recessive terms in [19]. For small $N$, the proof is straightforward and we have carried it out with the aid of computer algebra for $N \leq 26$.

The seven differential equations (5.9)–(5.15) each have only a single positive resonance $n_1$ associated with $a_0 = 1$. Hence the Laurent series (4.10) has only one arbitrary coefficient $a_{n_1}$ and no logarithms can appear. The $a_n$ are nonzero only when $n$ is a multiple of $n_1$. This solution has two parameters, the other being $x_0$. To proceed, we use the perturbative Painlevé test (the CFP test) of Conte, Fordy, and Pickering [9, 10]. We can terminate at the first perturbation order in the case of (5.9)–(5.13) and at the second perturbation order in the last two cases (5.14) and (5.15).

Because the differential equations are reduced, we can cast the CFP test in the form of a Painlevé $\alpha$-test:

$$x \;=\; x_0 + \alpha X, \tag{5.29}$$

$$y \;=\; \alpha^{-1} Y + Y_1 + \alpha Y_2 + \;\ldots\; . \tag{5.30}$$

The leading term $Y(X)$ satisfies the same reduced equation as $y(x)$, and we let $Y(X)$ be the two-parameter Laurent series solution. There is no loss of generality in translating the pole to $X = 0$, which means that we will be looking for conditions for the absence of logarithms in $Y_1(X)$ and/or $Y_2(X)$ at $X = 0$. Hence,

$$Y \;=\; a_0/X + a_1 + a_2 X + \;\ldots\; + a_n X^{n-1} + \;\ldots\;, \tag{5.31}$$

where $a_0 = 1$ and the only nonzero coefficients are those where $n$ is a multiple of the positive resonance number $n_1$.

The next term $Y_1(X)$ satisfies the homogeneous linear equation,

$$\begin{aligned} Y_1^{(\mathrm{iv})} &= AYY_1''' + \big(BY' + CY^2\big)Y_1'' \\ &\qquad + \big(BY'' + 2DYY' + EY^3\big)Y_1' \\ &\qquad + \big(AY''' + 2CYY'' + D(Y')^2 + 3EY^2Y' + 5FY^4\big)Y_1 \,, \end{aligned} \tag{5.32}$$

where the prime denotes $d/dX$. If $n_3$ denotes the negative resonance with greatest absolute value, then the Laurent expansion of $Y_1(X)$, at least if it is free of logarithms, will take the form,

$$Y_1 = \sum_{n=n_3}^{\infty} b_n X^{n-1} \,. \tag{5.33}$$

The values of $n$ at which logarithms may possibly arise in $Y_1$ are the resonances (in ascending order),

$$n = n_2,\ -1,\ n_1 \,, \tag{5.34}$$

and the coefficients $b_{n_3}$, $b_{n_2}$, $b_{-1}$, and $b_{n_1}$ are arbitrary. Observe that the formerly trivial resonance number $-1$ has been upgraded now to a more significant status, as pointed out by the originators of the CFP test.

For equation (5.9), having resonances $\{1, -2, -12\}$, we can proceed in two ways. One way is to start the series for $Y_1$ at the term $b_{-12}X^{-13}$ in accordance with (5.33). The first freedom-from logarithms condition occurs at the term $b_{-2}X^{-3}$ and the second at the next term $b_{-1}X^{-2}$. The latter contains the arbitrary constant $b_{-2}$ linearly, and the vanishing of the coefficient of $b_{-2}$ gives the relation,

$$D = \tfrac{9}{2}(B - 12). \tag{5.35}$$

With this simplification, the two conditions both become polynomial equations of degree 10 for $B$. The first has the single rational root $B = -120$, which does not satisfy the second. Hence equation (5.9) never has the Painlevé property.

A shorter calculation, which does not require computer algebra, can be done by starting the $Y_1$ series at the term $b_{-2}X^{-3}$. We lose a freedom-from-logarithms condition by doing this. The first such condition now occurs at the $b_{-1}X^{-2}$ term and is precisely (5.35). The second condition occurs at the constant term $b_1$. It is a cubic equation for $B$ with the single rational root $B = -20$. We have now arrived at the differential equation with coefficients,

$$\big\{A, B, C, D, E, F\big\} = \big\{-24, -20, -152, -144, -416, -128\big\}. \tag{5.36}$$

Three sets of resonance numbers are integers, but the fourth set, corresponding to $a_0 = 1/4$, has two irrational members. Hence we conclude again that (5.9) never has the Painlevé property.

For equation (5.10), having resonances $\{1, -3, -8\}$, we can also proceed in the same two ways. We choose the second and shorter way by starting the $Y_1$ series at the term $b_{-3}X^{-4}$. Conditions for the absence of logarithms occur at the terms $b_{-1}X^{-2}$ and $b_1$. These conditions are simultaneous polynomial equations for $B$ and $D$ that have the unique rational solution,

$$B = -30, \qquad D = -198.$$

We have arrived at the differential equation with coefficients,

$$\{A, B, C, D, E, F\} = \{-21, -30, -144, -198, -486, -162\}. \tag{5.37}$$

This equation admits a zero resonance number associated with the repeated root $a_0 = 1/3$ of (4.42). Hence (5.10) never has the Painlevé property.

For equation (5.11), having resonances $\{1, -4, -6\}$, the method where $Y_1$ begins with the term $b_{-6}X^{-7}$ is quicker. The first condition for the absence of logarithms, at the term $b_{-4}X^{-5}$, is quadratic in $D$ and splits into the two cases,

$$D = 6(B+4), \qquad D = \tfrac{1}{17}(107B + 554).$$

The second condition, at the term $b_{-1}X^{-2}$, contains the arbitrary coefficients $b_{-6}$ and $b_{-4}$. For each value of $D$, the coefficient of $b_{-4}$ gives a quadratic or cubic equation for $B$. None of the roots are rational. Hence equation (5.11) never has the Painlevé property.

For equation (5.12), having resonances $\{2, -2, -6\}$, the coefficients $a_n$ in the series (5.31) for $Y$ are nonzero only when $n$ is even. We let $Y_1$ begin with the term $b_{-6}X^{-7}$ in accordance with (5.33). The first condition for the absence of logarithms, at the term $b_{-2}X^{-3}$, is quadratic in $D$ and splits into the two cases,

$$D = 5B + 27, \qquad D = \tfrac{2}{11}(31B + 241).$$

The next condition, at the $b_{-1}X^{-2}$ term, is satisfied trivially. The third condition, at the $b_2X$ term, involves the arbitrary coefficients $b_{-6}$ and $b_{-2}$. For each value of $D$, the coefficient of $b_{-2}$ give a quadratic equation for $B$ having irrational roots only. Hence equation (5.12) never has the Painlevé property.

For equation (5.13), having resonances $\{2, -3, -4\}$, the coefficients $a_n$ are again nonzero only when $n$ is even. Start the $Y_1$ series at the term $b_{-4}X^{-5}$. The first condition for the absence of logarithms, at the term $b_{-3}X^{-4}$, is trivially satisfied. The next condition, at the term $b_{-1}X^{-2}$, is

$$D = \tfrac{1}{2}(9B+4).$$

The third condition, at the term $b_2X$, gives a cubic equation for $B$ that has no rational roots. Hence equation (5.13) never has the Painlevé property.

The final two equations (5.14) and (5.15) require that the CFP test be taken to the second perturbation order. In the case of (5.15), we also show an alternative $\alpha$-test that gives an immediate result. The function $Y_2(X)$ appearing in (5.30) satisfies the inhomogeneous linear differential equation,

$$\begin{aligned} Y_2^{(\mathrm{iv})} &- AYY_2''' - \big(BY' + CY^2\big)Y_2'' - \big(BY'' + 2DYY' + EY^3\big)Y_2' \\ &- \big(AY''' + 2CYY'' + D(Y')^2 + 3EY^2Y' + 5FY^4\big)Y_2 \\ &\quad = AY_1Y_1''' + BY_1'Y_1'' + 2CYY_1Y_1'' + DY(y_1')^2 \\ &\qquad + \big(2DY' + 3EY^2\big)Y_1Y_1' \\ &\qquad + \big(CY'' + 3EYY' + 10FY^3\big)(Y_1)^2. \end{aligned} \tag{5.38}$$

The series for $Y(X)$ and $Y_1(X)$ will take the forms (5.31) and (5.33), respectively, where $n_3$ once again denotes the negative resonance with largest absolute value. Then the series for $Y_2(X)$, at least if it is free of logarithms, will take the form,

$$Y_2 = \sum_{n=2n_3}^{\infty} c_n X^{n-1}. \tag{5.39}$$

The leading coefficient $c_{2n_3}$ is a constant multiple of $(b_{n_3})^2$. There are four possible conditions for the absence of logarithms. These correspond to the arbitrary coefficients $c_{n_3}$, $c_{n_2}$, $c_{-1}$, and $c_{n_1}$, in that order.

For equation (5.14), having resonances $\{3,-2,-4\}$, the coefficients $a_n$ in the series (5.31) for $Y$ are nonzero only when $n$ is a multiple of 3. We let $Y_1$ begin with the term $b_{-4}X^{-5}$. The only nontrivial condition for the absence of logarithms occurs at the term $b_{-1}X^{-2}$. It is

$$D = \tfrac{1}{2}(9B+52).$$

The series for $Y_2(X)$ begins at the term $c_{-8}X^{-9}$, where

$$c_{-8} = \tfrac{1}{176}(68-3B)(b_{-4})^2.$$

The freedom-from-logarithms condition at the term $c_{-4}X^{-5}$ forces $B = -20/3$. With this value of $B$, the next two conditions for the absence of logarithms, at the terms $c_{-2}X^{-3}$ and $c_{-1}X^{-2}$, are not satisfied. We conclude that equation (5.14) never has the Painlevé property.

For equation (5.15), having resonances $\{4, -2, -3\}$, the coefficients $a_n$ in the series (5.31) for $Y$ are nonzero only when $n$ is a multiple of 4. We let $Y_1$ begin with the term $b_{-3}X^{-4}$. All conditions for the absence of logarithms are trivially satisfied by the $Y_1$ series. The series for $Y_2(X)$ begins at the term $c_{-6}X^{-7}$, where

$$c_{-6} \;=\; \tfrac{1}{150}(D - 7B + 48)(b_{-3})^2.$$

The first condition for the absence of logarithms, at the $c_{-3}X^{-4}$ term, is identically satisfied. The second condition, at the term $c_{-2}X^{-3}$, is quadratic in $D$ and splits into the two cases,

$$D \;=\; 5B + 42, \qquad D \;=\; \tfrac{1}{23}(91B + 696). \tag{5.40}$$

The third condition, at the term $c_{-1}X^{-2}$, gives a quadratic equation for $B$. For the first value of $D$ in (5.40), the roots are $B = -30$ and $B = -240/23$. For the second value of $D$, the roots are $B = -40$ and $B = -240/29$. We have arrived at four differential equations, whose coefficients $\{A, B, C, D, E, F\}$ are

$$\big\{-12, -30, -54, -108, -108, 0\big\}, \qquad \big\{-12, -40, -64, -128, -128, 0\big\}, \tag{5.41}$$

$$\big\{-12, -240/23, -792/23, -234/23, 1116/23, 1350/23\big\},$$

$$\big\{-12, -240/29, -936/29, -72/29, 1728/29, 1800/29\big\}.$$

The last two equations fail the fourth condition at the term $c_4X^3$. A more direct violation in both cases is that some of the roots $a_{0i}$ of equation (4.42) are irrational. Equation (3.12) is contained in the class (5.15) up to a scaling, but it too has failed the CFP test. The two equations in (5.41), on the other hand, do have the Painlevé property. Except for a rescaling of the variable $y$, they are the reduced versions of equations F-XII and F-XIII, respectively.

Before presenting these two equations, we approach them from a different viewpoint. Let us set $D = 2(B - 24)$ in equation (5.15), which involves a loss of generality. If $B = 24$, the equation is equivalent under scaling to (3.12), which was shown to fail a Painlevé $\alpha$-test in Section 3. If $B \neq 24$, there are three sets of resonance numbers, all of which are integers except two that are algebraic functions of $B$. By setting

$$B \;=\; \frac{24(N^2 + 36)}{N^2 - 36}, \qquad D \;=\; \frac{3456}{N^2 - 36},$$

the remaining two resonance numbers become $N$ and $-N$, where $N$ is a positive integer different from 1, 4, or 6. After the scaling $y \to -y/6$, the differential equation becomes

$$y^{(\mathrm{iv})} = 2yy''' - 4y'y'' - \frac{16}{N^2-36}\left(6y'-y^2\right)\left(3y''-yy'\right). \tag{5.42}$$

It has the resonance numbers,

$$a_{01} = -6: \quad \{4,-2,-3\}, \qquad a_{02} = \tfrac{1}{2}(N-6): \quad \{1,4,N\},$$
$$a_{03} = -\tfrac{1}{2}(N+6): \quad \{1,4,-N\}.$$

Equation (5.42) admits the first integral,

$$y''' = 2yy'' - 3(y')^2 - \frac{4}{N^2-36}\left(6y'-y^2\right)^2 + K. \tag{5.43}$$

This is the Chazy-XII equation to which has been attached an additive constant. It is known that Chazy-XII only takes recessive terms when $N = 2$ and $N = 3$, and that an additive constant is an admissible recessive term in both cases. The corresponding members of the class (5.42) are the reduced equations F-XII and F-XIII, up to a scaling of $y$. For Chazy-XII, the recessive terms $A(x)y + B(x)$ when $N = 2$ and $B(x)$ when $N = 3$ were known to Chazy [6], but we do not know if Chazy was able to prove that recessive terms were not permitted for all $N \geq 4$ or for the limiting case $N = \infty$, which is the Chazy-III equation. The first published proof was given by Pickering [9] and Conte et al. [10] and was a triumphant application of their CFP test.

A neat alternative proof can be given using the Painlevé $\alpha$-test (5.29) and (5.30) up to the $Y_4(X)$ term, where the leading coefficient $Y(X)$ is the exact rational solution,

$$Y = \frac{N-6}{2(X-X_0)} - \frac{N+6}{2(X-X_1)}\,. \tag{5.44}$$

In the special case of (5.43), which is the relevant case here, the appropriate form of the $\alpha$-test is (3.13). Then $Y_1(X)$ satisfies a linear differential equation of the third order whose general solution is found to be

$$Y_1 = \left(\tfrac{1}{210}KI + K_2\right)(X-X_0)^{N-1}(X-X_1)^{-N-1} + \frac{K_3}{(X-X_0)^2} + \frac{K_4}{(X-X_1)^2}\,, \tag{5.45}$$

where

$$I = \int \left(\frac{X - X_1}{X - X_0}\right)^N \Big\{(X - X_0)^4 - 6(X - X_0)^3(X - X_1) + 15(X - X_0)^2(X - X_1)^2 - 6(X - X_0)(X - X_1)^3 + (X - X_1)^4\Big\}\, dX. \tag{5.46}$$

The integrand has a pole of order $N$ at $X = X_0$ with residue,

$$\tfrac{1}{24}N(N^2 - 4)(N^2 - 9)(X_0 - X_1)^5.$$

Hence $Y_1(X)$ contains a movable logarithmic branch point except when $N = 2$ and $N = 3$. (The case $N = 0$ is illegal because $N$ is a resonance number.) This method is easily extended to the general recessive term for Chazy-XII and Chazy-III involving seven arbitrary functions of $x$.

By suitably scaling $y$, we choose the following differential equations to be the standard forms of the reduced versions of equations F-XII and F-XIII:

$$\text{F-XII}: \quad y^{(\mathrm{iv})} = -4yy''' - 10y'y'' - 6y^2y'' - 12y(y')^2 - 4y^3y' = \frac{d}{dx}\Big\{ -4yy'' + 6(y')^2 - \big(3y' + y^2\big)^2\Big\}, \tag{5.47}$$

$$\text{F-XIII}: \quad y^{(\mathrm{iv})} = -3yy''' - 10y'y'' - 4y^2y'' - 8y(y')^2 - 2y^3y' = \frac{d}{dx}\Big\{ -3yy'' + \tfrac{9}{2}(y')^2 - \tfrac{1}{2}\big(4y' + y^2\big)^2\Big\}. \tag{5.48}$$

The scaling has been chosen so that a permutation of the $a_{0i}$ and their corresponding sets of resonance numbers places the case $a_{01} = 1$ with positive resonances only. The resonance data of F-XII are

$$a_{01} = 1: \ \{1, 2, 4\}, \qquad a_{02} = 2: \ \{1, -2, 4\}, \qquad a_{03} = 3: \ \{-2, -3, 4\}.$$

Similarly, the resonance data of F-XIII are

$$a_{01} = 1: \ \{1, 3, 4\}, \qquad a_{02} = 3: \ \{1, -3, 4\}, \qquad a_{03} = 4: \ \{-2, -3, 4\}.$$

In Section 7, we attach recessive terms to F-XII and F-XIII and give their solutions in terms of fourth-order linear differential equations.

The methods of this section do not reveal equations like (3.49) or (4.72) that have admissible integer resonances but fail conditions for the absence of logarithms. Out of curiosity, we carried out a computer search for such equations having $N_1 = 24$ with the search limits $|m_1| \leq 10$, $|m_2| \leq 20$, and $|m_3| \leq 50$ on the second set of resonances. The only examples found were derivatives of Chazy-XI with even $N \geq 8$ and Chazy-XII with $N = 5$ and $N \geq 7$. To these should be added equation (3.12) and the eight Martynov equations of the form (3.2), which have a single set of resonances.

This completes the analysis of the reduced equations having $N_1 = 24$, where $N_1$ is a product $n_1 n_2 n_3$ of resonance numbers. All cases with the Painlevé property belong to the family of equations (4.1) having $F = 0$. The reduced equations found were F-VII, F-VIII, F-X/XI, F-XII, and F-XIII. Except for the first two, they arise from the infinite family (4.19) of solutions of the Diophantine equation (4.18). No cases arose from the infinite family (4.53) of solutions of (4.51).

## 6. Cases with $A = 0$

This section carries out the Painlevé analysis of equations of the form (4.1) where $A = 0$. No other restrictions are imposed. The case $A = 0$ requires special handling because the other five coefficients $B, \ldots, F$ in the differential equation may be irrational and the leading coefficients $a_{0i}$ in the Laurent series (4.10) may be irrational, after normalizing one of them, say $a_{01} = 1$. Also, the argument that led to the simplifying constraint (4.21) for the case $F = 0$ is invalid when $A = 0$.

Three Painlevé-type reduced equations that have not already appeared above satisfy the hypotheses of this section, namely, F-XIV, F-XVII, and F-XVIII. Each are solvable in terms of hyperelliptic functions of genus 2. In the last two cases, the corresponding full equations define fourth-order Painlevé transcendents. They are related to fourth- and fifth-order equations having Bureau symbol $P2$ appearing in paper I.

Equations (3.41), (4.12)–(4.14), (4.29), (4.33), (4.43)–(4.46), (4.58), and (4.62) all imply, directly or indirectly, that

$$n_1 + n_2 + n_3 = 11, \qquad m_1 + m_2 + m_3 = 11, \tag{6.1}$$

$$p_1 + p_2 + p_3 = 11, \qquad q_1 + q_2 + q_3 = 11, \tag{6.2}$$

when $A = 0$, where $\{n_1, n_2, n_3\}$ denotes the resonance numbers associated with $a_{01}$, etc. If $F = 0$, there are fewer than four sets of resonance numbers, and so only three or fewer corresponding members of (6.1) and (6.2) are relevant, while the others should be ignored. We continue to use the notation,

$$N_1 = n_1 n_2 n_3, \qquad N_2 = m_1 m_2 m_3, \qquad N_3 = p_1 p_2 p_3, \qquad N_4 = q_1 q_2 q_3. \tag{6.3}$$

When there are four, three, or two sets of resonance numbers, the $N_i$ satisfy the Diophantine equations (4.51), (4.18), and (3.39), respectively. When there is just one set of resonance numbers, which is the case for equations (3.1) and (3.2), that set obeys $N_1 = 24$. It turns out that the constraints (6.1) and (6.2) on the factors of the $N_i$ are severe. For example, equation (3.39) has no solutions satisfying these constraints. That, in turn, implies that the infinite family (4.52) also has no solutions satisfying the constraints. On the other hand, the infinite families (4.19) and (4.53) have infinitely many solutions satisfying the constraints. These cases were fully analysed in the last section, including cases with $A = 0$, and we found that there were no Painlevé-type equations in these classes having $A = 0$. In fact the only equation having both $N_1 = 24$ and $A = 0$ is the reduced form of equation F-VII, given by (3.8) and (5.24).

The numbers of discrete solutions of (4.18) and (4.51) satisfying the above constraints on their factors are 32 and 874, respectively, where we have counted each $N_i$ as many times as the number of legal factorizations it admits. (For example, the solution, $N_1 = N_2 = 36$, $N_3 = N_4 = -144$, is counted six times because $-144$ admits three factorizations and there are six distinct pairings of factors for $N_3$ and $N_4$.) These lists are short enough to test each member for consistency with a differential equation. Only three distinct differential equations arise, namely, (6.32), (6.33), and (6.37) below. It is an inefficient method, however, because a substantial amount of computer time was needed to generate the list applying to (4.51) and because four sets of resonance numbers greatly overdetermine the differential equation.

A better method, at least when $N_1 \neq 24$, is the variant in Subsection 4.2, where one restricts attention (without loss of generality) to values of $N_1$ and $N_2$ subject to the bounds (4.54) and (4.55), as well as the constraints (6.1) and (6.2), regardless of whether or not such pairs are consistent with the Diophantine equation. There are exactly 90 pairs $\{N_1, N_2\}$ in range, counting multiplicities as above. For each pair, a set of coefficients $A, B, C, D, E, F$ expressed in terms of $a_{02}$ can be constructed from equations (4.58)–(4.68), except for (4.66) whose right-hand side degenerates to 0/0. Thence, by substituting these coefficients into (4.57) with $a_0 = a_{02}$, an equation for $a_{02}$ is found. The latter equation is cubic whether or not $F = 0$ and has three distinct roots in every case. Hence we have 270 differential equations to examine. All but 8 can be ruled out immediately because one or both of the cubic equations (4.45) and (4.46) for the third and fourth sets of resonance numbers (or just (4.45) only if $F = 0$) has noninteger coefficients. Three more equations can be ruled out because the resonances themselves are not integers. The surviving five equations contain three distinct members up to scaling, these being once again equations (6.32), (6.33), and (6.37) below. One advantage of doing the analysis by this method is that it rules out curiosities like equation (4.72) that have integer resonances but fail a test for the absence of logarithms.

Instead of the methods just described, we carry out the analysis of cases with $A = 0$ by the methods of the previous section, which are again very efficient. First, without loss of generality, we can choose $N_1$ to lie in the range $1 \leq N_1 \leq 96$, regardless of whether one or more coefficients of the differential equation are zero. An easy calculation shows that the only $N_1$ in range that have factors obeying the first member of (6.1) are

$$N_1 \;=\; 16, \;\; 21, \;\; 24, \;\; 36, \;\; 40, \;\; 96. \tag{6.4}$$

Furthermore, the admissible factorizations $\{n_1, n_2, n_3\}$ are unique in each case. Those factorizations are, respectively,

$$\{1,2,8\}, \qquad \{1,3,7\}, \qquad \{1,4,6\}, \tag{6.5}$$

$$\{2,3,6\}, \qquad \{2,4,5\}, \qquad \{-2,-3,16\}. \tag{6.6}$$

Except for the last case, the resonances are positive integers.

A single set of resonance numbers determines the differential equation up to two arbitrary constants according to (4.58)–(4.61). The six differential equations corresponding to the resonances (6.5) and (6.6) are

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & By'y'' + (B+20)y^2y'' + Dy(y')^2 \\ & + 2(D-B)y^3y' + (D-2B-16)y^5, \end{aligned} \tag{6.7}$$

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & By'y'' + (B+15)y^2y'' + Dy(y')^2 \\ & + (2D-2B-15)y^3y' + (D-2B-21)y^5, \end{aligned} \tag{6.8}$$

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & By'y'' + (B+12)y^2y'' + Dy(y')^2 \\ & + 2(D-B-12)y^3y' + (D-2B-24)y^5, \end{aligned} \tag{6.9}$$

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & By'y'' + (B+10)y^2y'' + Dy(y')^2 \\ & + 2(D-B-10)y^3y' + (D-2B-16)y^5, \end{aligned} \tag{6.10}$$

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & By'y'' + (B+8)y^2y'' + Dy(y')^2 \\ & + 2(D-B-12)y^3y' + (D-2B-16)y^5, \end{aligned} \tag{6.11}$$

$$
\begin{aligned}
y^{(\mathrm{iv})} \;=\;& By'y'' + (B+120)y^2y'' + Dy(y')^2 \\
& + 2(D-B+240)y^3y' + (D-2B+264)y^5. \qquad (6.12)
\end{aligned}
$$

For equation (6.7), which has resonances $\{1, 2, 8\}$, the condition for the absence of logarithms at the resonance 2 is

$$D \;=\; 3(B+10).$$

Thence, the condition at the resonance 8 splits into five polynomial equations, from cubic to sextic, for $B$. The only common solution is

$$B \;=\; -10.$$

Thus we have arrived at the differential equation with coefficients,

$$\big\{A, B, C, D, E, F\big\} \;=\; \big\{0, -10, 10, 0, 20, 4\big\}. \qquad (6.13)$$

This equation has three sets of integer resonances, but the fourth set, corresponding to $a_{04} = -1$, has two irrational members. Hence no equations of the form (6.7) have the Painlevé property.

For equation (6.8), which has resonances $\{1, 3, 7\}$, the condition for the absence of logarithms at the resonance 3 is quadratic in $D$ and splits into the two cases,

$$D \;=\; 3(B+10), \qquad D \;=\; \tfrac{5}{2}(B+11).$$

Thence the condition at the resonance 7 allows a single value of $B$ in each case, these values being

$$B \;=\; -9, \qquad B \;=\; -10,$$

respectively. Hence we have arrived at two differential equations with the coefficients,

$$\big\{0, -9, 6, 3, 9, 0\big\}, \qquad \big\{0, -10, 5, 5/2, 10, 3/2\big\}. \qquad (6.14)$$

The first of these equations has two sets of integer resonances, but the third set, corresponding to $a_{03} = -4/3$, is irrational. Similarly, the second of these equations has three sets of integer resonances, but the fourth set, corresponding to $a_{04} = -4/3$, has two irrational members. Hence no equations of the form (6.8) have the Painlevé property.

Equation (6.9), which has resonances $\{1, 4, 6\}$, has already been examined in the previous section. It is equation (5.7) and appears on the list of equations having $N_1 = 24$. We found that this case admits one and only one Painlevé-type equation, namely, (5.24), which is the reduced form of F-VII.

For equation (6.10), which has resonances $\{2,3,6\}$, only the last resonance $n_3 = 6$ gives a condition for the absence of logarithms. The condition is

$$D \;=\; B + 10,$$

and hence we arrive at the one-parameter family of differential equations,

$$y^{(\mathrm{iv})} \;=\; By'y'' + (B+10)(y^2y'' + y(y')^2) - (B+6)y^5. \tag{6.15}$$

There are several ways to proceed at this point. We show two methods that are quite different in style. The first is to examine the cubic equations (4.44)–(4.46) to get conditions for integer roots. The problem is complicated by the appearance of the undetermined parameter $B$ is the coefficients of (6.15). This method does not generalize well to higher-order problems, but is nevertheless instructive because it arises frequently in Painlevé classification problems. Part of the analysis in the previous section, for example, could have been done by this method. The second method, on the other hand, is very powerful and is easily generalized to higher-order problems.

For (6.15), the $a_0$ equation (4.42) has roots

$$a_{01} \;=\; 1, \qquad a_{02} \;=\; -2, \tag{6.16}$$

and, when $B \neq -6$,

$$a_{03} \;=\; \tfrac{1}{2}(1+r), \qquad a_{04} \;=\; \tfrac{1}{2}(1-r), \qquad r \;:=\; \sqrt{\frac{B+54}{B+6}}\,. \tag{6.17}$$

The cubic equation (4.44) for the second set of resonances $\{m_1, m_2, m_3\}$ admits the partial factorization,

$$(m-6)\big\{m^2 - 5m - 6(B+4)\big\} \;=\; 0. \tag{6.18}$$

This shows that $B$ is rational and, in particular, $B \neq -6$ because two of the roots would be irrational otherwise. Hence

$$m_1 \;=\; 6, \qquad m_2 = N+2, \qquad m_3 \;=\; 3-N, \tag{6.19}$$

$$B \;=\; \tfrac{1}{6}(N+5)(N-6), \qquad r \;=\; \sqrt{\frac{N^2-N+294}{N^2-N+6}}\,, \tag{6.20}$$

where $N$ is a positive integer, $N \neq 3,\ 4$. From (6.17) and the coefficients of the cubic equations (4.45) and (4.46), we see that $a_{03}$, $a_{04}$, and $r$ must also be rational. It follows that the right-hand side of the equation,

$$M^2 \;=\; (N^2-N+6)(N^2-N+294), \tag{6.21}$$

must be the square of an integer, and so this equation is a Diophantine equation for positive integers $M$ and $N$. It can be rewritten

$$M^2 = (N^2 - N + 150)^2 - 12^4.$$

An easy calculation produces the impossible statement,

$$N^2 - N + 149 < M < N^2 - N + 150,$$

whenever $N \geq 102$. Hence $1 \leq N \leq 101$. The only values of $N$ in range that satisfy (6.21) are

$$N = 1,\ 3,\ 6,\ 10,\ 13,\ 23. \tag{6.22}$$

Of these, $N = 13$ and $N = 23$ can be discarded because the coefficients of (4.45) and (4.46) are not integers and $N = 3$ is inadmissible because it gives a zero resonance number. The differential equation derived from the case $N = 10$ is just a rescaling of the differential equation derived from $N = 1$. Hence, we have narrowed the investigation down to two cases,

$$N = 1, \qquad N = 6. \tag{6.23}$$

These will lead to the reduced equations F-XVIII and F-XVII, respectively.

Before writing these equations down, we demonstrate the second method for analysing equation (6.15). In this case $N_1 = 36$. According to (4.55), there is no loss of generality in restricting $N_2$ to the range,

$$36 \leq N_2 \leq 216.$$

The only $N_2$ in range that have factorizations $\{m_1, m_2, m_3\}$ obeying the second member of (6.1) are

$$N_2 = 36,\ 40,\ 96,\ 136,\ 180,\ 216, \tag{6.24}$$

and the factorizations in each case are unique. The first three factorizations are given above by (6.6) and the last three by

$$\{-2, -4, 17\}, \qquad \{-2, -5, 18\}, \qquad \{-3, -4, 18\}. \tag{6.25}$$

When the coefficients of equation (6.15) are introduced into (4.63) and (4.64), we get

$$(B + 10)(a_{02})^2 = Ba_{02} + 7Q_2 - R_2 - 31, \tag{6.26}$$

$$(B + 10)(a_{02})^2 = Ba_{02} - 5Q_2 + 2R_2 - \tfrac{1}{2}N_2 + 11, \tag{6.27}$$

where $Q_2 = 11$ and $R_2$ is defined by (4.26). An easy calculation shows that (6.26) and (6.27) are compatible only in the case $N_2 = 36$, in which case,

$$(B+10)(a_{02})^2 \; = \; Ba_{02} + 10. \tag{6.28}$$

But $a_{02}$ cannot equal 1 and so $B \neq -10$ and

$$a_{02} \; = \; -\frac{10}{B+10}\,. \tag{6.29}$$

But $a_{02}$ must also be one of the roots of equation (4.57), which simplifies to

$$(B+6)(a_{02})^3 + (B+6)(a_{02})^2 - 2(B+12)a_{02} - 24 \; = \; 0. \tag{6.30}$$

The only values of $B$ that satisfy (6.29) and (6.30) are

$$B \; = \; 0, \;\; -5, \;\; 10. \tag{6.31}$$

We have arrived at three differential equations of the form (6.15), of which two are distinct up to scaling. When $B = 0$ and $B = -5$, we get the reduced forms of equations F-XVII and F-XVIII, respectively. These are given by

$$\text{F-XVII}: \qquad y^{(\mathrm{iv})} \; = \; 10y^2y'' + 10y(y')^2 - 6y^5, \tag{6.32}$$

$$\text{F-XVIII}: \qquad y^{(\mathrm{iv})} \; = \; -\,5y'y'' + 5y^2y'' + 5y(y')^2 - y^5. \tag{6.33}$$

The case $B = 10$ is a rescaling of (6.33). The resonance data for F-XVII are

$$a_{01} = 1: \;\; \{2,3,6\}, \qquad a_{02} = -1: \;\; \{2,3,6\},$$
$$a_{03} = 2: \;\; \{-3,6,8\}, \qquad a_{04} = -2: \;\; \{-3,6,8\}.$$

The resonance data for F-XVIII are

$$a_{01} = 1: \;\; \{2,3,6\}, \qquad a_{02} = -2: \;\; \{2,3,6\},$$
$$a_{03} = -3: \;\; \{-2,6,7\}, \qquad a_{04} = 4: \;\; \{6,-7,12\}.$$

Equation (6.32) was found by Martynov [21], who also added some recessive terms. The full equation can be deduced from a member of a hierarchy of equations containing antiderivatives given by Ablowitz and Segur [44] and is written out explicitly by Flaschka and Newell [45], who also gave the hierarchy. This hierarchy is one of two different hierarchies of ordinary differential equations that goes by the name of the Painlevé-II hierarchy and can be obtained by group-invariant reduction of the

Modified KdV hierarchy. Equation F-V, given by (1.8), is the corresponding member of the Painlevé-I hierarchy deduced by reduction of the KdV hierarchy.

Equation F-XVIII, on the other hand, has not been found in previous attempts at the Painlevé classification of fourth-order equations, e.g., [21, 25]. The corresponding full equation also defines a fourth-order Painlevé transcendent and is a member of a hierarchy. It can be obtained by group-invariant reduction of the Modified Sawada-Kotera equation (4.4). The next section contains the full versions of F-XVII and F-XVIII and their relations to equations appearing in paper I, as well as particular solutions involving classical Painlevé transcendents and hyperelliptic functions.

For equation (6.11), which has resonances $\{2, 4, 5\}$, only the middle resonance $n_2 = 4$ gives a condition for the absence of logarithms. The condition is

$$D = 2(B + 8), \tag{6.34}$$

and so we have arrived at the one-parameter family of differential equations,

$$\begin{aligned} y^{(\mathrm{iv})} &= By'y'' + (B+8)y^2y'' + 2(B+8)y(y')^2 + 2(B+4)y^3y', \\ &= \frac{d}{dx}\Big\{\tfrac{1}{2}B(y')^2 + (B+8)y^2y' + \tfrac{1}{2}(B+4)y^4\Big\}. \end{aligned} \tag{6.35}$$

The equation has been reduced to the derivative of a third-order equation in the Chazy polynomial class. From the analysis in [6, 8] or by the methods shown above, we conclude that Painlevé-type equations occur only for the cases,

$$B = 4\big(1 \pm \sqrt{5}\,\big). \tag{6.36}$$

These two cases are equivalent to each other under a scaling and can be given rational coefficients by the rescaling,

$$y \to -\tfrac{3}{2}\big(1 \mp \sqrt{5}\,\big)y.$$

Hence, we take the standard form of the reduced equation F-XIV to be

$$\text{F-XIV}: \qquad y^{(\mathrm{iv})} = 24y'y'' + 72y^2y'' + 144y(y')^2 + 216y^3y'. \tag{6.37}$$

Its resonance data are

$$a_{01} = 1: \ \{-3, 4, 10\}, \qquad a_{02} = \tfrac{1}{6}\big(1 + \sqrt{5}\,\big): \ \{2, 4, 5\},$$
$$a_{03} = \tfrac{1}{6}\big(1 - \sqrt{5}\,\big): \ \{2, 4, 5\}.$$

A first integral of F-XIV is

$$y''' = 12(y')^2 + 72y^2y' + 54y^4 + \delta, \tag{6.38}$$

which is the Chazy-IX equation, and a second integral and the general solution is given in [19]. It is known [6] that the only nontrivial recessive term that the Chazy-IX equation can take is an additive constant. It does not automatically follow that F-XIV takes no recessive terms, but an easy calculation shows the latter to be true. The solution of F-XIV in terms of hyperelliptic functions of genus 2 is given in the next section.

The final equation to be tested is (6.12), whose resonances are $\{-2, -3, 16\}$. The negative resonances suggest that we should attack this case with the CFP test, as we did in Section 5, but this is actually quite a difficult example to test with the CFP test. Fortunately, if $N_1 = 96$, then either

$$\{N_1, N_2, N_3, N_4\} = \{96, 96, 96, 96\} \tag{6.39}$$

or else one of the $N_i$ lies in the range $1 \le N_i \le 95$, in which case it would have already been examined in this section. Hence we only need to examine the case,

$$B = 0, \qquad D = -240,$$

implied by (6.39). The differential equation (6.12) becomes

$$y^{(\mathrm{iv})} = 120y^2y'' - 240y(y')^2 + 24y^5. \tag{6.40}$$

The resonances corresponding to $a_0 = \pm 1$ are $\{-2, -3, 16\}$ while the resonances corresponding to $a_0 = \pm i$ are irrational. Hence equation (6.40) does not have the Painlevé property and neither does any member of the family (6.12).

This concludes the Painlevé analysis of the class of fourth-order reduced equations (4.1) where not all coefficients are zero. Exactly 11 distinct equations of this form have the Painlevé property. They are denoted F-VII, ..., F-XVIII with the labels F-X and F-XI applying to the same reduced equation. They are given above by (3.8), (3.9), (3.47), (5.26), (5.47), (5.48), (6.37), (4.40), (4.70), (6.32), and (6.33), respectively. We also gave the number F-XIX to the Clarkson-Olver equation (4.9), which violates the Painlevé property in a dramatic fashion but is nevertheless integrable according to our usage of the term. In addition, we found a number of non-Painlevé equations having integer resonances, the most interesting being (4.7).

## 7. The full equations F-VII–F-XIX and their solutions

In this section, we construct the full versions of equations F-VII, ..., F-XVIII and give the general solutions of those that do not define fourth-order Painlevé transcendents.

In the latter cases, F-XVII and F-XVIII, we give particular solutions involving classical Painlevé transcendents and hyperelliptic functions and discuss their relationships with equations appearing in paper I. We also give the solution of the Clarkson-Olver equation F-XIX and include a brief discussion of higher-order generalizations, including one that involves a Painlevé-I transcendent.

The reduced equations are

$$\text{F-VII}: \qquad y^{(\mathrm{iv})} = -12y'y''; \tag{7.1}$$

$$\text{F-VIII}: \qquad y^{(\mathrm{iv})} = -2yy''' - 6y'y''; \tag{7.2}$$

$$\text{F-IX}: \qquad y^{(\mathrm{iv})} = -3yy''' - 9y'y'' - 3y^2y'' - 6y(y')^2; \tag{7.3}$$

$$\text{F-X/XI}: \qquad y^{(\mathrm{iv})} = -2yy''' - 6y'y'' + \frac{48}{N^2-1}\left(y' + y^2\right)\left(y'' + 2yy'\right); \tag{7.4}$$

$$\text{F-XII}: \qquad y^{(\mathrm{iv})} = -4yy''' - 10y'y'' - 6y^2y'' - 12y(y')^2 - 4y^3y'; \tag{7.5}$$

$$\text{F-XIII}: \qquad y^{(\mathrm{iv})} = -3yy''' - 10y'y'' - 4y^2y'' - 8y(y')^2 - 2y^3y'; \tag{7.6}$$

$$\text{F-XIV}: \qquad y^{(\mathrm{iv})} = 24y'y'' + 72y^2y'' + 144y(y')^2 + 216y^3y'; \tag{7.7}$$

$$\text{F-XV}: \qquad y^{(\mathrm{iv})} = -2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y'; \tag{7.8}$$

$$\text{F-XVI}: \qquad y^{(\mathrm{iv})} = -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5; \tag{7.9}$$

$$\text{F-XVII}: \qquad y^{(\mathrm{iv})} = 10y^2y'' + 10y(y')^2 - 6y^5; \tag{7.10}$$

$$\text{F-XVIII}: \qquad y^{(\mathrm{iv})} = -5y'y'' + 5y^2y'' + 5y(y')^2 - y^5; \tag{7.11}$$

$$\begin{aligned}\text{F-XIX}: \qquad y^{(\mathrm{iv})} = {} & \tfrac{10}{3}yy''' - 4y'y'' - \tfrac{8}{3}y^2y'' + 4y(y')^2 \\ & + 4\alpha\left(6y' - y^2\right)\left(3y'' - 3yy' + \tfrac{1}{3}y^3\right).\end{aligned} \tag{7.12}$$

Except for F-XIX, which will not be assigned any recessive terms, the corresponding full versions of these equations will take the form,

$$\begin{aligned}y^{(\mathrm{iv})} = {} & A(x)yy''' + B(x)y'y'' + C(x)y^2y'' + D(x)y(y')^2 \\ & + E(x)y^3y' + F(x)y^5 + G(x, y, y', y'', y'''),\end{aligned} \tag{7.13}$$

where $G(x, y, y', y'', y''')$ collects the recessive terms given by

$$\begin{aligned} G(x, y, y', y'', y''') &= G(x)y''' + H(x)yy'' + I(x)(y')^2 + J(x)y^2y' \\ &\quad + \tilde{K}(x)y^4 + L(x)y'' + M(x)yy' + N(x)y^3 \\ &\quad + P(x)y' + Q(x)y^2 + R(x)y + S(x). \end{aligned} \tag{7.14}$$

Thus we have 12 functions, $G(x)$, ..., $S(x)$, to determine such that equation (7.13) has the Painlevé property. (We placed a tilde on the symbol $K$ because we want to use $K$ with or without a subscript, as well as other symbols, as an integration constant.) We may use scaling freedom to make the first six coefficients $A$, ..., $F$ constant and equal to their counterparts in the reduced equations. However, in some cases, we prefer a gauge in which the first six coefficients are not necessarily constant, in which case there will be some scaling function $\sigma(x)$ such that

$$\begin{aligned} A(x) &= A\sigma, \qquad B(x) = B\sigma, \qquad C(x) = C\sigma^2, \\ D(x) &= D\sigma^2, \qquad E(x) = E\sigma^3, \qquad F(x) = F\sigma^4. \end{aligned}$$

The gauge freedom is

$$y = \lambda(x)\bar{y} + \mu(x), \qquad \bar{x} = \phi(x). \tag{7.15}$$

If the first six coefficients are to be held constant, then

$$\lambda(x) = \phi'(x). \tag{7.16}$$

When deriving admissible recessive terms, we use the gauge freedom to remove two coefficients or place two suitable constraints on the coefficients. There are no strict rules for choosing gauge. A guiding principle is to keep the early terms of a Laurent series of the form (4.10) as simple as possible, at least for one of the values of $a_0$. In particular, if 1 is not a resonance number, we often choose gauge to set $a_1 = 0$. However, having constructed all the recessive terms subject to a specific gauge choice, one is free to change or relax the gauge before presenting a canonical form for the equation.

In most cases, the construction of the recessive terms follows well-established procedures. The method based on Laurent series, that was first used extensively by Chazy and extended by Bureau, is sufficient for our purposes in this section. Good references for this method are, for example, Bureau [2], Ablowitz et al. [46], Ablowitz and Clarkson [47], and Conte et al. [10]. The latter reference also contains the CFP test, which is a substantial generalization. The earlier method of $\alpha$-expansions invented by Painlevé [26, pp. 224–225] and reviewed by Ince [27, pp. 329–330] appears to be more cumbersome, but is able to handle many difficult cases that are out of reach of the Laurent-series method, such as the problem of recessive terms for Chazy-III or proving the non-Painlevé character of the Bureau barrier equation F-II. Many CFP tests can be cast in the form of Painlevé $\alpha$-expansions in which each term is a Laurent series. In Sections 3 and 5 above, CFP tests and $\alpha$-expansions were essential tools. But here, we have enough positive resonances to use the Laurent series methods and consequently we can be very brief about the details.

### *7.1. Equation F-VII, derivative of Chazy-I and SD-I.*

Recessive terms for F-VII were calculated by Bureau [8, pp. 95–106]. However, we would like to present an alternative choice of gauge that allows a ready reduction to the six classical Painlevé equations. This form of F-VII will have rational coefficients and will admit the version [19, (A.3)] of the Chazy-I equation as a first integral and the version [36, (5.1)] of the second-order second-degree equation SD-I as a second integral. We will refer the reader to the latter reference for the explicit reductions to the differential equations governing the six classical Painlevé transcendents.

The first step is to prove that the full F-VII equation is a derivative of the full Chazy-I equation. The equation under investigation is

$$y^{(\mathrm{iv})} \;=\; -\,12y'y'' + G(x, y, y', y'', y'''), \tag{7.17}$$

where $G(x, y, y', y'', y''')$ is given by (7.14). There is only one set of resonance numbers:

$$a_0 \;=\; 1: \;\; \{n_1, n_2, n_3\} \;=\; \{1, 4, 6\}.$$

Since 1 is a resonance number, the coefficient $a_1$ in the Laurent series (4.10) is an arbitrary constant of integration. A choice of gauge that simplifies the coefficient $a_2$ is

$$H - J + 2\tilde{K} \;=\; 0, \qquad 2L - M + N \;=\; 0. \tag{7.18}$$

The choice of gauge favored by Bureau is $H = 0$ instead of the first member of (7.18). Both choices lead to the same canonical form for F-VII.

The condition for the absence of logarithms at the first resonance $n_1 = 1$ is

$$\tilde{K} \;=\; 6G - 2H - I + J. \tag{7.19}$$

The condition at the second resonance $n_2 = 4$ is cubic in $a_1$ and therefore gives four separate constraints. The condition at the third resonance $n_3 = 6$ is quartic in $a_1$ and linear in $a_4$ and gives seven more constraints. Taking the simplest constraints first and combining them with (7.18) and (7.19), we obtain

$$H \;=\; 0, \qquad I \;=\; 6G, \qquad J \;=\; 0, \qquad \tilde{K} \;=\; 0. \tag{7.20}$$

The remaining constraints now simplify considerably. After suitable rearrangement, they yield a third-order equation for $G(x)$,

$$G''' \;=\; -\,2GG'' - 2(G')^2 + 3\big(G' + G^2\big)^2, \tag{7.21}$$

which is the reduced Chazy-XI equation with $N = 3$. It can be integrated to give a Riccati equation,

$$G' + G^2 \;=\; 2a(x), \tag{7.22}$$

where the auxiliary variable $a(x)$ satisfies

$$a'' \;=\; 6a^2, \qquad (a')^2 \;=\; 4a^3 + \gamma^2, \tag{7.23}$$

$\gamma$ constant. The Riccati equation can be linearized according to

$$G(x) \;=\; h'(x)/h(x), \qquad h'' \;=\; 2ah. \tag{7.24}$$

Without loss of generality, $a(x)$ is one of the following:

$$a(x) \;=\; \wp(x; 0, -\gamma^2), \qquad a(x) \;=\; 1/x^2, \qquad a(x) \;=\; 0. \tag{7.25}$$

In the first case, F-VII is solvable in terms of Painlevé-VI functions. In the second case, it is solvable in terms of Painlevé-V or Painlevé-III functions. In the third case, it is solvable in terms of Painlevé-IV, Painlevé-II, or Painlevé-I functions. All cases admit subcases solvable with elliptic or simpler functions. The respective forms of $h(x)$ are

$$h(x) \;=\; \alpha\, a^{1/2} \exp\Big\{\tfrac{1}{2}\gamma \int (1/a)\, dx\Big\} \;+\; \beta\, a^{1/2} \exp\Big\{-\tfrac{1}{2}\gamma \int (1/a)\, dx\Big\},$$

$$h(x) \;=\; \alpha x^2 + \frac{\beta}{x}, \qquad h(x) \;=\; \alpha x + \beta.$$

where $\alpha$ and $\beta$ are constants, not both zero. Alternative forms for $a(x)$ and $h(x)$ in terms of a parameter $\bar{x}$ are given below.

The remaining coefficients in (7.14) can all be constructed in terms of $a(x)$ and $h(x)$ by quadratures. The result is the full F-VII equation in Bureau's gauge:

$$\begin{aligned}
\text{F-VII}: \qquad y^{(\mathrm{iv})} \;=\; & -\,12y'y'' + \frac{d}{dx}\Big\{ -\,6(y')^2 + 6a(x)\big(y' + y^2\big) \\
& + 6b(x)y\Big\} \;+\; G(x)\Big\{y''' + 6(y')^2 \\
& -\,6a(x)\big(y' + y^2\big) - 6b(x)y\Big\} \;+\; S(x),
\end{aligned} \tag{7.26}$$

where $b(x)$ and $S(x)$ satisfy

$$b'' = 6ab, \qquad S' + GS = 6b^2. \tag{7.27}$$

When written in this form, it is easy to see that F-VII admits the Chazy-I equation as a first integral. The Chazy-I equation appears in the gauge chosen by Chazy [6, 48] and Bureau [8] and is given by

$$\text{Chazy-I}: \qquad y''' = -6(y')^2 + 6a(x)\big(y' + y^2\big) + 6b(x)y + 6c(x), \tag{7.28}$$

where $c(x)$ contains the first integration constant $K_1$ and is found by solving

$$c' - Gc = \tfrac{1}{6}S. \tag{7.29}$$

Elimination of $G(x)$ from (7.27) and (7.29) gives the differential constraint,

$$c'' = 2ac + b^2, \tag{7.30}$$

which completes the identification of (7.28) with the Chazy-I equation.

Equation (7.28) admits the first integral,

$$\begin{aligned}\text{SD-I}: \qquad (y'' - 2ay - b)^2 = & -4(y')^3 + 12ay^2y' - 4a'y^3 + 4a(y')^2 \\ & + (12b - 4a')yy' + (4a^2 - 6b')y^2 \\ & + (12c - 2b')y' + (4ab - 12c')y \\ & + b^2 - 12\int bc\,dx + K_2,\end{aligned} \tag{7.31}$$

$K_2$ being the second integration constant, which is a differential equation of the second order and second degree [6, 19, 36, 49, 50, 51]. It can be put in the binomial form [36, 52],

$$(\bar{y}'')^2 = F(\bar{x}, \bar{y}, \bar{y}'), \tag{7.32}$$

by a gauge transformation of the form (7.31) where $\lambda(x)$, $\mu(x)$, and $\phi(x)$ satisfy

$$\lambda'' = 2a\lambda, \qquad \mu'' = 2a\mu + b, \qquad \lambda\phi'' + 2\lambda'\phi' = 0. \tag{7.33}$$

These equations can be all be solved in terms of $a(x)$ and $h(x)$ by quadratures. The solutions of the first and third equations are

$$\lambda(x) = k_1h + k_2h\int\frac{dx}{h^2}, \qquad \phi(x) = \frac{k_3 + k_4\int(1/h^2)\,dx}{k_1 + k_2\int(1/h^2)\,dx}, \tag{7.34}$$

where the $k_i$ are constant with $k_1k_4 - k_2k_3 \neq 0$. There is no loss of generality in setting

$$k_1 \;=\; k_4 \;=\; 1, \qquad k_2 \;=\; k_3 \;=\; 0.$$

We can rewrite the differential equations for $a(x)$ and $h(x)$ above in terms of the new independent variable,

$$\bar{x} \;=\; \int \frac{dx}{h^2}\,. \tag{7.35}$$

We find that $h$ satisfies the fourth-order equation,

$$\begin{aligned} h^3\,\frac{d^4h}{d\bar{x}^4} - 16h^2\,\frac{dh}{d\bar{x}}\,\frac{d^3h}{d\bar{x}^3} - 12h^2\left(\frac{d^2h}{d\bar{x}^2}\right)^2 \\ + 120h\left(\frac{dh}{d\bar{x}}\right)^2\frac{d^2h}{d\bar{x}^2} - 120\left(\frac{dh}{d\bar{x}}\right)^4 \;=\; 0, \end{aligned}$$

whose general solution is

$$h \;=\; \big\{f(\bar{x})\big\}^{-1/3}, \qquad f(\bar{x}) \;:=\; c_1\bar{x}^3 + c_2\bar{x}^2 + c_3\bar{x} + c_4\,. \tag{7.36}$$

The $c_i$, where $i$ will run from 1 to 16 below, are constant, with the first four being not all zero. Thence

$$a \;=\; \tfrac{1}{9}f^{-2/3}\Big\{\big(c_2^2 - 3c_1c_3\big)\bar{x}^2 + \big(c_2c_3 - 9c_1c_4\big)\bar{x} + \big(c_3^2 - 3c_2c_4\big)\Big\}. \tag{7.37}$$

When taken together with the genus-1 abelian integral,

$$x \;=\; \int \big\{f(\bar{x})\big\}^{-2/3}\,d\bar{x}, \tag{7.38}$$

these results provide an alternative way so solve the $a$ and $h$ equations (7.23) and (7.24). The expressions for $b$, $c$, and $\mu$ in terms of $\bar{x}$ are a little lengthier and involve a quadrature, but when they are substituted into equations (7.26), (7.28), and (7.31), they yield elegant equations with rational coefficients.

We present the results of those calculations. Drop the bars on $x$ and $y$. Equations

F-VII, Chazy-I, and SD-I become

$$
\begin{aligned}
\text{F-VII :} \qquad y^{(\mathrm{iv})} \;&=\; -\,\frac{3f'}{f}\,y''' \;-\; \frac{ff''+(f')^2}{f^2}\,y'' \\
&\quad -\,\frac{4}{f^2}\Big\{3c_1x^2(xy'-y) + c_2x(3xy'-2y) \\
&\quad + c_3(3xy'-y) + 3c_4y' + c_5x^2 + c_6x + c_7\Big\}y'' \\
&\quad -\,\frac{2}{f^2}\Big\{3c_1(xy'-y)^2 + 2c_2y'(xy'-y) + c_3(y')^2 \\
&\quad + 2c_5(xy'-y) + c_6y' + c_8\Big\},
\end{aligned}
\tag{7.39}
$$

$$
\begin{aligned}
\text{Chazy-I :} \qquad y''' \;&=\; -\,\frac{f'}{f}\,y'' \;-\; \frac{2}{f^2}\Big\{3c_1x(xy'-y)^2 \\
&\quad + c_2(xy'-y)(3xy'-y) + c_3y'(3xy'-2y) \\
&\quad + 3c_4(y')^2 + 2c_5x(xy'-y) + c_6(2xy'-y) \\
&\quad + 2c_7y' + c_8x + c_9\Big\};
\end{aligned}
\tag{7.40}
$$

$$
\begin{aligned}
\text{SD-I :} \qquad (y'')^2 \;&=\; -\,\frac{4}{f^2}\Big\{c_1(xy'-y)^3 + c_2y'(xy'-y)^2 + c_3(y')^2(xy'-y) \\
&\quad + c_4(y')^3 + c_5(xy'-y)^2 + c_6y'(xy'-y) \\
&\quad + c_7(y')^2 + c_8(xy'-y) + c_9y' + c_{10}\Big\},
\end{aligned}
\tag{7.41}
$$

where $f(x) = c_1x^3 + c_2x^2 + c_3x + c_4$. We choose (7.39) to be the standard form for equation F-VII. Equations (7.40) and (7.41) are already the standard forms that we have chosen for Chazy-I and SD-I, respectively [19, 36]. Differentiation of SD-I to eliminate the parameter $c_{10}$ yields Chazy-I; differentiation of Chazy-I to eliminate the parameter $c_9$ yields F-VII.

There is some redundancy in these equations. Equations Chazy-I and SD-I are invariant under changes of variable of the form,

$$
\bar{x} \;=\; \frac{a_1x + a_2}{a_3x + a_4}, \qquad \bar{y} \;=\; \frac{a_5y + a_6x + a_7}{a_3x + a_4}, \tag{7.42}
$$

which form a group of transformations. This freedom can be used to normalize $f(x)$ to one of the three forms,

$$
f(x) \;=\; x(x-1), \qquad f(x) \;=\; x, \qquad f(x) \;=\; 1, \tag{7.43}
$$

which correspond to the three respective forms of $a(x)$ in (7.25). In these forms Chazy-I and SD-I are directly transformable into the six differential equations governing the classical Painlevé transcendents. Because all the Painlevé transcendents are included in a single equation, we call SD-I the "master Painlevé equation." We refer the reader to [36] for the explicit transformations.

Now the chosen form for F-VII is not invariant under (7.42) unless $a_3 = 0$. This prevents the normalization of $f(x)$ before the equation has been integrated up to Chazy-I. The full transformation group can be admitted if the gauge of F-VII is suitably relaxed. This can be done by letting the coefficients $k_i$ in (7.34) be arbitrary. The resulting equation takes the form,

$$\text{F-VII}: \qquad \begin{aligned} y^{(\mathrm{iv})} &= \frac{fg' - 3f'g}{fg}\, y''' - \frac{ff''g + (f')^2 g - ff'g'}{f^2 g}\, y'' \\ &\quad - \frac{2}{f^2}\left\{\frac{d}{dx} - \frac{g'}{g}\right\}\Big\{3c_1 x(xy'-y)^2 \\ &\quad + c_2(xy'-y)(3xy'-y) + c_3 y'(3xy'-2y) \\ &\quad + 3c_4(y')^2 + 2c_5 x(xy'-y) + c_6(2xy'-y) \\ &\quad + 2c_7 y' + c_8 x + c_9\Big\}, \end{aligned} \tag{7.44}$$

where $f(x)$ is as above and $g(x) := d_1 x + d_2$. This is the form presented in the Appendix of paper I. It can be deduced by adding $Kg/f^2$ to the right-hand side of (7.40) (which amounts to a renaming of the parameters $c_8$ and $c_9$) and then differentiating in such a way as to eliminate the constant $K$.

There is an enormous number of higher-order Painlevé-type equations that can be obtained from SD-I by differentiation. One of the most spectacular is the following rational third-order equation:

$$y''' = -\frac{f'}{f}\, y'' - \frac{2R_1}{f^2} + \frac{S_1}{2S}\left\{\left(y''\right)^2 + \frac{4R}{f^2}\right\}, \tag{7.45}$$

where $f(x)$ is as above and

$$\begin{aligned} R := {} & c_1(xy'-y)^3 + c_2 y'(xy'-y)^2 + c_3(y')^2(xy'-y) \\ & + c_4(y')^3 + c_5(xy'-y)^2 + c_6 y'(xy'-y) \\ & + c_7(y')^2 + c_8(xy'-y) + c_9 y' + c_{10}\,, \end{aligned} \tag{7.46}$$

$$\begin{aligned}
R_1 &:= 3c_1x(xy'-y)^2 + c_2(xy'-y)(3xy'-y) + c_3y'(3xy'-2y) \\
&\quad + 3c_4(y')^2 + 2c_5x(xy'-y) + c_6(2xy'-y) \\
&\quad + 2c_7y' + c_8x + c_9\,, && (7.47) \\
S &:= c_{11}(xy'-y)^2 + c_{12}y'(xy'-y) + c_{13}(y')^2 \\
&\quad + c_{14}(xy'-y) + c_{15}y' + c_{16}\,, && (7.48) \\
S_1 &:= 2c_{11}x(xy'-y) + c_{12}(2xy'-y) + 2c_{13}y' + c_{14}x + c_{15}\,. && (7.49)
\end{aligned}$$

Equation (7.45), which we call the "third-order master Painlevé equation," is built upon the fundamental reduced equation,

$$Y''' = \frac{1}{2}\left\{\frac{1}{Y'} + \frac{1}{Y'-1}\right\}(Y'')^2, \tag{7.50}$$

which is one of thirteen such reduced equations. It can be deduced from SD-I by replacing $c_i$ by $c_i + Kc_{i+6}$ for $i = 5, 6, \ldots, 10$ and then differentiating so as to eliminate $K$. Hence the generic equation of the form (7.45) can be solved in terms of Painlevé-VI functions and every equation of this form can be solved in terms of one of the six classical Painlevé transcendents. A particular member of this class, solvable in terms of Painlevé-V, has arisen in a recent combinatorics problem of Tracy and Widom [53, 54].

### *7.2. Equation F-VIII, derivative of Chazy-II and Ince-V.*

Recessive terms for F-VIII were calculated by Bureau [8, pp. 106–109]. This case is easy because the resonance numbers are low, but there is one interesting twist.

The equation under investigation is

$$y^{(\mathrm{iv})} = -2yy''' - 6y'y'' + G(x, y, y', y'', y'''), \tag{7.51}$$

where $G(x, y, y', y'', y''')$ is given by (7.14). There is only one set of resonance numbers:

$$a_0 = 1: \quad \{n_1, n_2, n_3\} = \{2, 3, 4\}.$$

A gauge choice that sets $a_1 = 0$ is

$$\tilde{K} = 6G - 2H - I + J.$$

Thence the three resonances give six constraints, leaving five arbitrary coefficient functions. A second gauge condition could be imposed, but there is no advantage in doing so. The resulting differential equation is

$$\begin{aligned} y^{(\mathrm{iv})} \;=\;& -2yy''' - 6y'y'' + A(x)\{y''' + 2yy'' + 2(y')^2\} \\ &+ H(x)\big(y' + y^2\big)^2 + B(x)\big(y'' + 2yy'\big) + C(x)\big(y' + y^2\big) + D(x), \end{aligned} \tag{7.52}$$

where the coefficients are arbitrary functions. (Note that the symbols chosen here for coefficient functions do not correspond to similar symbols appearing in (7.13) and (7.14). We will be renaming coefficient functions as convenience dictates throughout this section.)

Equation (7.52) has one spurious coefficient which the resonance analysis failed to identify. To see this, let

$$w \;=\; y' + y^2. \tag{7.53}$$

Then (7.52) simplifies to the third-order equation,

$$w''' \;=\; A(x)w'' + H(x)w^2 + B(x)w' + C(x)w + D(x), \tag{7.54}$$

which must have the Painlevé property whenever (7.52) does so. The Painlevé $\alpha$-test,

$$x \;=\; x_0 + \alpha X, \qquad w \;=\; \alpha^{-3}W,$$

yields the reduced equation,

$$W''' \;=\; H(x_0)W^2, \tag{7.55}$$

where the prime denotes $d/dX$. If $H(x_0) \neq 0$, this equation has Bureau symbol $P3$ and is known to have irrational (nonreal) resonances. Since $x_0$ is arbitrary, we must have $H(x) = 0$.

The full version of equation F-VIII is

$$\begin{aligned} \text{F-VIII}: \qquad y^{(\mathrm{iv})} \;=\;& -2yy''' - 6y'y'' + A(x)\{y''' + 2yy'' + 2(y')^2\} \\ &+ B(x)\big(y'' + 2yy'\big) + C(x)\big(y' + y^2\big) + D(x). \end{aligned} \tag{7.56}$$

It is solved in terms of a third-order linear equation and a Riccati equation by setting $H(x) = 0$ in (7.53) and (7.54). This equation was presented in a more general gauge in paper I, but is still solvable in the same fashion. By taking successive first integrals of the $w$-equation, F-VIII integrates to Chazy-II, then to Ince-V, and then to a Riccati equation. As pointed out by Exton [25], equations of this type generalize trivially to all differential orders.

### *7.3. Equation F-IX, derivative of Chazy-IV and Ince-VI.*

Recessive terms for F-IX were calculated by Bureau [8, pp. 109–116]. This case is also easy because the resonance numbers are low. The equation under investigation is

$$y^{(\mathrm{iv})} \;=\; -\,3yy''' - 9y'y'' - 3y^2y'' - 6y(y')^2 + G(x,y,y',y'',y'''). \tag{7.57}$$

There are two sets of resonance numbers:

$$a_{01} = 1: \quad \{1,3,4\}, \qquad a_{02} = 2: \quad \{-2,3,4\}.$$

A choice of gauge that simplifies the coefficient $a_2$ in the series with $a_0 = 1$ is

$$2\tilde{K} - J + H \;=\; 0, \qquad N - M + 2L \;=\; 0.$$

The first set of resonances gives seven constraints, leaving three arbitrary coefficient functions. With coefficient functions suitably renamed, the resulting differential equation is

$$\text{F-IX}: \qquad \begin{aligned} y^{(\mathrm{iv})} \;=\; &-\,3yy''' - 9y'y'' - 3y^2y'' - 6y(y')^2 \\ &+ A(x)\bigl\{y''' + 3yy'' + 3(y')^2 + 3y^2y'\bigr\} \\ &+ B(x)\bigl\{y'' + 3yy' + y^3\bigr\} + C(x). \end{aligned} \tag{7.58}$$

We take this to be the standard form of equation F-IX. Because it is necessary to solve a Riccati equation for $\mu(x)$ to impose the second gauge condition, it is useful to have F-IX in a more general gauge. With no gauge restrictions, the equation takes the form,

$$\text{F-IX}: \qquad \begin{aligned} y^{(\mathrm{iv})} \;=\; &-\frac{d^2}{dx^2}\Bigl\{3yy' + y^3 + P(x)(y'+y^2) + Q(x)y + R(x)\Bigr\} \\ &+ A(x)\frac{d}{dx}\Bigl\{y'' + 3yy' + y^3 + P(x)(y'+y^2) \\ &+ Q(x)y + R(x)\Bigr\} + B(x)\Bigl\{y'' + 3yy' + y^3 \\ &+ P(x)(y'+y^2) + Q(x)y + R(x)\Bigr\} + C(x), \end{aligned} \tag{7.59}$$

which is the form given in paper I. When $P = Q = R = 0$, equation (7.59) reduces to (7.58). The function $R(x)$ is redundant because it can be absorbed into $C(x)$.

The latter form makes the solution transparent. Let

$$w \;:=\; y'' + 3yy' + y^3 + P(x)(y'+y^2) + Q(x)y + R(x). \tag{7.60}$$

Then (7.59) simplifies to the linear equation,

$$w'' = A(x)w' + B(x)w + C(x). \tag{7.61}$$

The nonlinear equation (7.60) for $y$ is one of the equations found by Painlevé along with his more celebrated transcendent equations P-I, P-II, and P-III [55]. It was given the number (5) by Gambier [29] and VI by Ince [27, Chapter 14], and we call it Ince-VI. It can be reduced to a third-order linear equation by

$$y = \frac{u'}{u}, \qquad u''' + P(x)u'' + Q(x)u' + \big\{R(x) - w\big\}u = 0. \tag{7.62}$$

By taking successive integrals of the $w$-equation, F-IX integrates to the Chazy-IV equation and then to Ince-VI. Like F-VIII, this equation also generalizes trivially to all differential orders.

### *7.4. Equations F-X and F-XI, nested Riccati and Chazy-XIII.*

Equations F-X and F-XI have the same reduced equation (7.4). This reduced equation, which is a derivative of the reduced Chazy-XI equation, has the Painlevé property when $N = 2$ and when $N$ is an odd integer $\geq 3$, but is integrable in terms of Lamé functions for unrestricted complex $N$, except for $N = \pm 1$ where the coefficients are infinite. The case $N = \pm 1$ can be included by a straightforward limit and becomes F-VII, while the case $N = \infty$ is the F-VIII equation. We exclude these limiting cases from the present discussion. The full equations F-X and F-XI will turn out to be more general than derivatives of full Chazy-XI equations and will involve the first, second, and fourth Painlevé transcendents.

The full equations under investigation both take the form,

$$\begin{aligned} y^{(\mathrm{iv})} = &- 2yy''' - 6y'y'' + \frac{48}{N^2 - 1}\big(y' + y^2\big)\big(y'' + 2yy'\big) \\ &+ G(x, y, y', y'', y'''). \end{aligned} \tag{7.63}$$

There are three sets of resonance numbers:

$$a_{01} = 1 : \ \{2, 3, 4\}, \qquad a_{02} = \tfrac{1}{2}(1 - N) : \ \{4, 6, N\},$$
$$a_{03} = \tfrac{1}{2}(1 + N) : \ \{4, 6, -N\}.$$

Since the parameter $N$ is a resonance number, it must be a nonzero integer (positive, without loss of generality) for (7.63) to have the Painlevé property. Also the cases $N = 1$, 4, and 6 are excluded because of infinite coefficients or repeated resonances.

For the case $a_0 = 1$, the coefficient $a_1$ in the Laurent series (4.10) can be set to zero by the gauge choice,

$$\tilde{K} \;=\; 6G - 2H - I + J.$$

Thence the resonances $\{n_1, n_2, n_3\} = \{2, 3, 4\}$ give the constraints,

$$H \;=\; 2G, \qquad J \;=\; 2(I - 2G), \qquad \tilde{K} \;=\; I - 2G,$$
$$M \;=\; 2L, \qquad N \;=\; 0, \qquad Q \;=\; P, \qquad R \;=\; 0.$$

For the two cases $a_0 = (1 \pm N)/2$, the coefficient $a_1$ can be set to zero by the gauge choice,

$$I \;=\; \frac{2(N^2 - 13)}{N^2 - 1}\, G,$$

which can be done legally for all complex values of $N$ except the excluded values $\pm 1$. All the remaining constraints can be deduced from the case $a_0 = (1-N)/2$, where the resonances are $\{4, 6, N\}$. From this point onwards, the admissible cases with $N \leq 6$ need to be treated individually and are postponed to a few paragraphs ahead. For $N \geq 7$, the resonance number 4 gives the constraint,

$$P' - GP \;=\; L'' - 2GL' - \bigl(G' - G^2\bigr)L.$$

Thence the resonance number 6 gives two constraints, the simpler of which is

$$G' + G^2 \;=\; 0. \tag{7.64}$$

This equation admits two solutions,

$$G(x) \;=\; 1/x, \qquad G(x) \;=\; 0, \tag{7.65}$$

which lead to qualitatively different full equations. The equations will be denoted F-X and F-XI, respectively. Equation F-X is new while equation F-XI (with incorrect recessive terms) was found by Exton [25].

When $G(x) = 1/x$, the remaining constraints obtained from the resonance number 6 lead, after some rearrangement, to the following form of equation F-X:

$$\text{F-X}: \qquad y^{(\mathrm{iv})} \;=\; \frac{d}{dx} R(x, y, y', y'') \;+\; \frac{1}{x}\Bigl\{y''' - R(x, y, y', y'')\Bigr\}$$
$$+\; \lambda x\bigl\{y' + y^2 - g(x)\bigr\} - \tfrac{1}{96}(N^2 - 1)\lambda^2 x^3, \tag{7.66}$$

where $\lambda$, $\alpha$, and $\beta$ are arbitrary complex constants, $g(x)$ is an arbitrary locally analytic function, and

$$R(x,y,y',y'') := -2yy'' - 2(y')^2 + g''(x) + \tfrac{1}{4}(N^2-1)(\alpha x+\beta) + [24/(N^2-1)]\{y' + y^2 - g(x)\}^2. \tag{7.67}$$

Similarly, when $G(x) = 0$, the remaining constraints obtained from the resonance number 6 lead to the following form of equation F-XI:

$$\text{F-XI}: \qquad y^{(\mathrm{iv})} = \frac{d}{dx}R(x,y,y',y'') + \lambda\{y' + y^2 - g(x)\} - \tfrac{1}{24}(N^2-1)\lambda^2 x, \tag{7.68}$$

where $R(x,y,y',y'')$ is again given by (7.67). Note that F-X contains the parameter $\beta$ but not $\alpha$, while F-XI contains the parameter $\alpha$ but not $\beta$. If $\lambda = 0$, both F-X and F-XI admit the Chazy-XI equation,

$$y''' = R(x,y,y',y''), \tag{7.69}$$

as a first integral in which $\alpha$ and $\beta$ are the respective constants of integration.

Although, these forms of F-X and F-XI were derived on the assumption that $N \geq 7$, they will be seen below to be applicable to the cases $N = 2$, 3, and 5, which also contain Painlevé-type equations. In fact, we will take (7.66) and (7.68) to be the definitions of F-X and F-XI, respectively, when $N$ is an unrestricted complex number (except $N = \pm 1$), because these equations are already integrable. When $N$ is an integer, the solutions of F-X and F-XI, in general, admit movable branch points because we have not applied the conditions for the absence of logarithms at the resonance $N$. These additional constraints will be examined below.

The solution of F-X is given by nesting a Riccati equation with the generic form of the Chazy-XIII equation, denoted Chazy-XIII.a in [19]. Let $w(x)$ be the general solution of the Chazy-XIII.a equation,

$$w''' = 12ww' + \frac{1}{x}\left(w'' - 6w^2\right) + \lambda xw - \tfrac{1}{24}\lambda^2 x^3 - \frac{\beta}{x}. \tag{7.70}$$

Then the general solution of F-X is given by solving the Riccati equation,

$$y' = -y^2 + g(x) + \tfrac{1}{4}(N^2-1)w. \tag{7.71}$$

When $\lambda \neq 0$, the solution of equation (7.70) is given in terms of the Painlevé-IV equation,

$$\text{P-IV}: \qquad u'' = \frac{1}{2u}(u')^2 - \tfrac{3}{2}\lambda u^3 - 2\lambda xu^2 - (\tfrac{1}{2}\lambda x^2 + \gamma)u + \frac{\delta}{u}, \tag{7.72}$$

by

$$w \;=\; \tfrac{1}{2}\lambda_1 u' - \tfrac{1}{2}\lambda(u^2 + xu + \tfrac{1}{6}x^2) - \tfrac{1}{6}(\gamma - \lambda_1), \tag{7.73}$$

where $\gamma$ is a constant of integration and the other parameters $\lambda_1$ and $\delta$ appearing in (7.72) and (7.73) are defined by

$$\lambda_1 := \pm\sqrt{-\lambda}\,, \qquad \delta := -(1/\lambda)\big\{\beta + \tfrac{1}{6}\gamma(\gamma - 2\lambda_1)\big\}. \tag{7.74}$$

When $\lambda = 0$, equation (7.70) can be integrated directly to give the Painlevé-I equation,

$$\text{P-I}: \qquad w'' \;=\; 6w^2 + \gamma x + \beta, \tag{7.75}$$

where $\gamma$ is again the constant of integration.

In a similar fashion, the solution of F-XI is given by nesting a Riccati equation with the nongeneric form of the Chazy-XIII equation, denoted Chazy-XIII.b in [19]. The Riccati equation is (7.71) above, while the function $w(x)$ on the right-hand side satisfies the Chazy-XIII.b equation,

$$w''' \;=\; 12ww' + \lambda w - \tfrac{1}{6}\lambda^2 x + \alpha. \tag{7.76}$$

When $\lambda \neq 0$, the solution of equation (7.76) is given by

$$w \;=\; \tfrac{1}{2}(\pm u' + u^2) + \tfrac{1}{12}(\lambda x - 6\alpha/\lambda), \tag{7.77}$$

where $u(x)$ satisfies the Painlevé-II equation,

$$\text{P-II}: \qquad u'' \;=\; 2u^3 + (\lambda x - 6\alpha/\lambda)u + \delta. \tag{7.78}$$

When $\lambda = 0$, (7.76) has the Painlevé-I first integral,

$$\text{P-I}: \qquad w'' \;=\; 6w^2 + \alpha x + \delta. \tag{7.79}$$

In (7.78) and (7.79), $\delta$ is the constant of integration.

The solutions of F-X and F-XI just given involve solving a Riccati equation containing an additive meromorphic function $w(x)$ expressible in terms of the classical Painlevé transcendents I, II, or IV, or simpler functions in certain cases. In general, the solution of the Riccati equation admits movable logarithmic branch points at the double poles of $w(x)$.

Now we impose the strict Painlevé property and include the cases $N = 2$, 3, and 5, which were set aside during the analysis of the second and third sets of resonance numbers above. Note that, in these three cases as well as surviving cases with $N \geq 7$, equations F-X and F-XI will still take the forms (7.66) and (7.68), respectively, but there will be additional constraints on the parameters $\alpha$, $\beta$, and $\lambda$ and the function $g(x)$.

Consider, first, equation F-X. It turns out that F-X has the strict Painlevé property only in the cases,

$$N = 2, 3, 7.$$

When $N = 2$ and $N = 3$, the resonance $N$ gives the constraints,

$$N = 2: \qquad g(x) = 0; \tag{7.80}$$

$$N = 3: \qquad g(x) = -\tfrac{1}{12}\lambda x^2 + g_0, \tag{7.81}$$

where $g_0$ is a constant. In both these cases, the parameters $\lambda$ and $\beta$ are unrestricted. Hence both cases admit solutions involving Painlevé-I and Painlevé-IV. When $N = 7$, there are three possibilities:

$$g(x) = 0, \qquad \lambda = 0, \qquad \beta \text{ unrestricted;} \tag{7.82}$$

$$g(x) = \frac{15}{4x^2}, \qquad \lambda = \beta = 0; \tag{7.83}$$

$$g(x) = \frac{30}{x^2}, \qquad \lambda = \beta = 0. \tag{7.84}$$

These cases admit solutions involving the Painlevé-I function and are derivatives of special cases of Chazy-XI with $N = 7$. The latter equation allows a much richer class of coefficient functions $g(x)$ because $g(x)$ is allowed to depend on $\alpha$ in Chazy-XI but not in F-X. In fact, the $g(x)$ for Chazy-XI with $N = 7$ is the Fif-I transcendent of paper I, which is related to the transcendent defined by F-XVIII (see below).

Equation F-XI has the strict Painlevé property in the interesting cases,

$$N = 2, 3, 5, 7, \tag{7.85}$$

as well as for all odd $N \geq 9$ subject to the restrictions that $\lambda = 0$, $\alpha = 0$, and $g(x)$ is constant. The latter cases, which are just derivatives of Chazy-XI, the latter being severely restricted in a similar fashion except for $N = 11$, can be solved in terms of Lamé functions. In the four interesting cases, $\lambda$ and $\alpha$ are unrestricted and solutions involving Painlevé-I and Painlevé-II are admitted. These cases are given by

$$N = 2: \qquad g(x) = 0; \tag{7.86}$$

$$N = 3: \qquad g(x) = -\tfrac{1}{6}\lambda x + g_0; \tag{7.87}$$

$$N = 5: \qquad g''' = 2gg' + \lambda g - \lambda^2 x + 6\alpha; \tag{7.88}$$

$$N = 7: \qquad g(x) = 0. \tag{7.89}$$

Exton [25] found the special case $\lambda = 0$ of (7.86) and the Lamé function cases, but missed the remaining cases. When $N = 7$, there is a second case with constant $g(x)$ belonging to the aforementioned Lamé function family. Members of this family are also included in (7.87) and (7.88). The most interesting case is $N = 5$, where $g(x)$ satisfies another Chazy-XIII.b equation, a rescaling of (7.76). When $\lambda \neq 0$, it is solved in terms of Painlevé-II by

$$\begin{aligned} g &= 3(\pm h' + h^2) + \tfrac{1}{2}\lambda x - 3\alpha/\lambda, \\ h'' &= 2h^3 + (\lambda x - 6\alpha/\lambda)h + h_0 , \end{aligned}$$

where $h_0$ is constant. When $\lambda = 0$, $g(x)$ satisfies the same Painlevé-I equation as it does in Chazy-XI with $N = 5$ [19].

It is clear that equations Chazy-XI, F-X, and F-XI can be generalized in a number of directions. Any Painlevé-type equation with the Bureau symbol $P2$ for a function $w(x)$ can be nested with a Riccati equation of the form (7.71). For example, a number of interesting fifth-order equations can be deduced in this way from F-I and F-III, ..., F-VI. One such example is given by

$$N = 5, \qquad g(x) = -\tfrac{1}{3}\alpha, \qquad w(x) \text{ satisfies F-VI with } \kappa = 0, \tag{7.90}$$

the latter equation (see (1.9)) having a complicated solution in terms of hyperelliptic functions found in paper I. Also, the role of the Riccati equation can be played by any other Painlevé-type equation that is able to take arbitrary coefficient functions, there being numerous examples in the Painlevé-Gambier class and other lists. The most interesting cases are where $w(x)$ or $g(x)$ or both, or functions playing a similar role, are Painlevé transcendents, higher-order transcendents, or hyperelliptic or abelian functions. Particular cases can be deduced trivially from the associated linear problems admitted by the transcendents, but Chazy-XI with $N = 7$ and F-XI with $N = 5$ are examples of more interesting constructions of this type.

### *7.5. Equations F-XII and F-XIII, derivatives of Chazy-XII.*

The reduced forms of F-XII and F-XIII are equivalent under scaling to the special cases $N = 2$ and $N = 3$ of equation (5.42), which is the derivative of the Chazy-XII equation. It is well known [6, 10] that the Chazy-XII equation can take recessive terms only for these two values of $N$. It is a straightforward calculation to show that full versions of F-XII and F-XIII are also just derivatives of the full versions of Chazy-XII for $N = 2$ and $N = 3$. These equations were found by Exton [25] and Martynov [21].

The equations under consideration take the forms,

$$y^{(\mathrm{iv})} = -4yy''' - 10y'y'' - 6y^2y'' - 12y(y')^2 - 4y^3y' + G(x, y, y', y'', y'''); \quad (7.91)$$

$$y^{(\mathrm{iv})} = -3yy''' - 10y'y'' - 4y^2y'' - 8y(y')^2 - 2y^3y' + G(x, y, y', y'', y'''). \quad (7.92)$$

The respective resonance data are

$$a_{01} = 1: \quad \{1, 2, 4\}, \qquad a_{02} = 2: \quad \{1, -2, 4\}, \qquad a_{03} = 3: \quad \{-2, -3, 4\};$$

$$a_{01} = 1: \quad \{1, 3, 4\}, \qquad a_{02} = 3: \quad \{1, -3, 4\}, \qquad a_{03} = 4: \quad \{-2, -3, 4\}.$$

Exton and Martynov chose the following gauge:

$$G = 0, \qquad H = 0.$$

Resonance analysis gives seven conditions for the absence of logarithms in F-XII and eight in F-XIII.

After suitably renaming the coefficient functions, the full versions of F-XII and F-XIII are found to be

$$\begin{aligned} \text{F-XII}: \qquad y^{(\mathrm{iv})} &= -4yy''' - 10y'y'' - 6y^2y'' - 12y(y')^2 - 4y^3y' \\ &\quad + A(x)\{y'' + 2yy'\} + \{B(x) + A'(x)\}y' \\ &\quad + A'(x)y^2 + B'(x)y + C(x); \end{aligned} \quad (7.93)$$

$$\begin{aligned} \text{F-XIII}: \qquad y^{(\mathrm{iv})} &= -3yy''' - 10y'y'' - 4y^2y'' - 8y(y')^2 - 2y^3y' \\ &\quad + 2A(x)\{y'' + yy'\} + A'(x)\{3y' + y^2\} \\ &\quad + A''(x)y + C(x), \end{aligned} \quad (7.94)$$

where $A(x)$, $B(x)$, and $C(x)$ are arbitrary functions. These equations are equivalent under scaling to the versions presented in the Appendix of paper I, the latter forms being chosen to highlight the connection with Chazy-XII. Equations (7.93) and (7.94) admit the respective first integrals,

$$\begin{aligned} y''' &= -4yy'' - 3(y')^2 - 6y^2y' - y^4 \\ &\quad + A(x)\{y' + y^2\} + B(x)y + D(x), \end{aligned} \quad (7.95)$$

$$\begin{aligned} y''' &= -3yy'' - \tfrac{7}{2}(y')^2 - 4y^2y' - \tfrac{1}{2}y^4 \\ &\quad + A(x)\{2y' + y^2\} + A'(x)y + D(x), \end{aligned} \quad (7.96)$$

where, in both cases,

$$D(x) := \int C(x)\,dx + K_1\,,$$

where $K_1$ is the constant of integration.

Equations F-XII and F-XIII are both solvable by reduction to fourth-order linear differential equations. The general solution of (7.95) is given by

$$y \;=\; \frac{w'}{w}\,, \qquad w^{(\mathrm{iv})} \;=\; A(x)w'' + B(x)w' + D(x)w. \tag{7.97}$$

The general solution of (7.96) is given by

$$y \;=\; \frac{w_1 w_2'' - w_1'' w_2}{w_1 w_2' - w_1' w_2}\,, \tag{7.98}$$

where $w_1(x)$ and $w_2(x)$ are two linearly independent solutions of the equation,

$$w^{(\mathrm{iv})} \;=\; Aw'' + A'w' + \tfrac{1}{2}\big\{A'' - \tfrac{1}{2}A^2 - D\big\}w, \tag{7.99}$$

subject to the constraint,

$$w_1 w_2''' - w_1' w_2'' + w_1'' w_2' - w_1''' w_2 - A(x)\big\{w_1 w_2' - w_1' w_2\big\} \;=\; 0, \tag{7.100}$$

which reduces to a single constraint among the constants of integration in $w_1$ and $w_2$. The denominator on the right-hand side of (7.98) satisfies a fifth-order linear equation, which is the usual way to present the solution of a Chazy-XII equation with $N = 3$.

### *7.6. Equation F-XIV, derivative of Chazy-IX.*

We will show that equation F-XIV takes no recessive terms, except those trivially induced by a change of gauge. Hence F-XIV is the derivative of the Chazy-IX equation, the latter taking an additive constant term. Equation F-XIV was found by Martynov [21] and Exton [25, p. 398], but the latter author rejected the equation because of an incorrect belief that it did not have the Painlevé property.

Chazy Classes IX and X are distinguished among the thirteen equations in the Chazy polynomial class by having solutions with irrational residues at the movable poles. They were not solved until recently [19]. The solutions in each case involve hyperelliptic functions of genus 2.

The class of equations under investigation is

$$y^{(\mathrm{iv})} \;=\; 24y'y'' + 72y^2y'' + 144y(y')^2 + 216y^3y' + G(x,y,y',y'',y'''). \tag{7.101}$$

The resonance data are

$$a_{01} = 1 : \ \{-3, 4, 10\}, \qquad a_{02} = \tfrac{1}{6}\big(1+\sqrt{5}\,\big) : \ \{2, 4, 5\},$$
$$a_{03} = \tfrac{1}{6}\big(1-\sqrt{5}\,\big) : \ \{2, 4, 5\}.$$

We examine the conditions for the absence of logarithms implied by the second and third sets of resonances together. A choice of gauge that sets $a_1 = 0$ in both the corresponding Laurent series is

$$J = -3(36G + 2H + I), \qquad \tilde{K} = -9(18G + 2H + I).$$

The resonance 2 gives the constraints $M = -6L$ and $N = -18L$. Thence the coefficients of $a_4$ and $(a_2)^2$ in the conditions at resonance 5 give $G = H = I = 0$. Finally, the remaining conditions at the resonances 4 and 5 force all the recessive terms to vanish.

The full equation F-XIV is

$$\text{F-XIV}: \qquad y^{(\text{iv})} = 24y'y'' + 72y^2y'' + 144y(y')^2 + 216y^3y'. \tag{7.102}$$

A first integral is the Chazy-IX equation,

$$\text{Chazy-IX}: \qquad y''' = 12(y')^2 + 72y^2y' + 54y^4 + \delta, \tag{7.103}$$

where $\delta$ is the constant of integration, and a less obvious second integral is

$$\begin{aligned} y(y'')^3 \; &- \; \Big\{\tfrac{1}{2}(y')^2 + \tfrac{1}{3}(54y^4+\delta)\Big\}(y'')^2 \\ &- \; \Big\{12y(y')^3 + 72y^3(y')^2 + 3y(54y^4+\delta)y'\Big\}y'' \\ &+ \; \tfrac{24}{5}(y')^5 + 72y^2(y')^4 + 4(108y^4+\delta)(y')^3 \\ &+ 24y^2(54y^4+\delta)(y')^2 \; + \; \tfrac{2}{3}(54y^4+\delta)^2y' \\ &+ \; y^2\big(\tfrac{4374}{5}y^8 + 54\delta y^4 + \tfrac{3}{2}\delta^2\big) \; = \; K, \end{aligned} \tag{7.104}$$

where $K$ is the second constant of integration.

In [19], Chazy-IX is solved in terms of hyperelliptic functions by transforming to a particular case of F-V, given by (1.8), with $\kappa = 0$ and employing Drach's classical solution of the latter [13]. We refer the reader to paper I and [19] for the details of this construction. Here, we present the result.

Let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_{\infty}^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_{\infty}^{v(x)} \frac{dt}{\sqrt{P(t)}} = C_1 \,, \tag{7.105}$$

$$\int_{\infty}^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_{\infty}^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = x + C_2 \,, \tag{7.106}$$

where $P(t)$ is the quintic polynomial,

$$P(t) := t^5 - 2\alpha t^3 + \tfrac{4}{5}\alpha^2 t + 16K_2 \,, \tag{7.107}$$

with coefficients,

$$\alpha = -6(3+\sqrt{5}\,)\delta, \qquad K_2 = \tfrac{324}{25}(25+11\sqrt{5}\,)K \,. \tag{7.108}$$

The general solution of F-XIV and Chazy-IX is

$$y = \frac{\frac{1}{12}\left(\sqrt{5}\,-1\right)(u-v)(u'-v')}{u^2+v^2+\frac{1}{2}\left(\sqrt{5}\,+1\right)uv+\frac{1}{5}\left(\sqrt{5}\,-5\right)\alpha} \,. \tag{7.109}$$

The constants $C_1$ and $C_2$ are the third and fourth constants of integration. (The usage of symbols here is consistent with the usage in [19].) Changing the sign of $\sqrt{5}$ gives two distinct presentations of the solution, from which can be deduced a discrete transformation of F-XIV:

$$\bar{y} = \tfrac{1}{6}\left(1-\sqrt{5}\,\right)\frac{y''' - 3(1-\sqrt{5}\,)\left\{\frac{1}{2}yy'' + (y')^2\right\} - 9(3-\sqrt{5}\,)y^2y'}{y'' - 3(1-\sqrt{5}\,)yy'} \,, \tag{7.110}$$

$$\bar{\delta} = (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}\,)^4\delta, \qquad \bar{K} = (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}\,)^{10}K. \tag{7.111}$$

The solution (7.109) is globally meromorphic. It can be verified by direct substitution (with the aid of computer algebra) into any of the differential equations (7.102)–(7.104). The calculation can be shortened by using the auxiliary variable $w(x)$ given by [19, (3.7)] that satisfies F-V. For particular values of the ratio $K/\delta^{5/2}$, the solution is expressible in terms of elliptic functions and quadratures or two elliptic functions with different periods. For $K = \delta = 0$, the solution is a rational function given by [19, (3.22)].

The Chazy-IX and Chazy-X equations are the most mysterious and interesting of the 13 equations in the Chazy polynomial class. Chazy himself conjectured that they would be solved in terms of known functions and suggested elliptic functions as a possibility. (This is correct in the case of the reduced form of Chazy-X.) Exton

[25, p. 398] cast doubt on this by claiming that both equations violated the Painlevé property. We now know that the solutions of both equations are globally meromorphic and therefore the equations themselves do have the Painlevé property. The solution of Chazy-X in [19] is remarkably intricate. It was obtained by transforming Chazy-X into a special case of F-VI, the autonomous case of which was solved in terms of hyperelliptic functions in paper I.

The derivative of the Chazy-X equation does not appear among the fourth-order Painlevé-type equations because Chazy-X does not take an additive constant and because the derivative has the repeated resonance number 4. The full Chazy-X equation (first given in [19]) is

$$\begin{aligned} y''' &= 6y^2y' + \tfrac{3}{11}(9+7s)(y'+y^2)^2 - \tfrac{1}{22}(4-3s)\alpha y' \\ &\qquad + \tfrac{1}{44}(3-5s)\alpha y^2 - \tfrac{1}{352}(9+7s)\alpha^2, \end{aligned} \tag{7.112}$$

where $s := \pm\sqrt{3}$. This equation has a complicated first integral of the second order and fourth degree [19, (4.6)]. If the parameter $\alpha$ is eliminated by differentiation, the result is the following fourth-order differential equation of the second degree:

$$\begin{aligned} &\Big\{y^{(\mathrm{iv})} - \tfrac{20}{3}sy'y'' - 2(7+s)y^2y'' - 4(7+s)y(y')^2 - 2(3+5s)y^3y'\Big\}^2 \\ &= \tfrac{4}{33}(81-47s)\Big\{y'' - (3+s)yy'\Big\}^2 \\ &\qquad \times \Big\{y''' - \tfrac{10}{3}s(y')^2 - 2(7+s)y^2y' - \tfrac{1}{2}(3+5s)y^4\Big\}. \end{aligned} \tag{7.113}$$

This is an example of an "exotic Painlevé equation" as defined in [41]. Its general integral is globally meromorphic while its 3-parameter singular integral has movable branch points. The singular integral is obtained by setting the last factor on the right-hand side of (7.113) to zero. The latter differential equation has the same shape as Chazy-IX and Chazy-X but has two sets of irrational resonance numbers. (The exoticness here is of a milder type than in the cases discussed in [41] because the Bureau symbol for (7.113) is not fractional but is $P1$ as for Chazy-X.)

### *7.7. Equation F-XV, generalization of Chazy-VI.*

Equation F-XV is a new equation that was discovered by the methods of Subsection 4.1 above. The corresponding full equation to be found takes the form,

$$y^{(\mathrm{iv})} = -2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y' + G(x,y,y',y'',y'''). \tag{7.114}$$

The resonance data are

$$a_{01} = 1: \ \{1,3,5\}, \qquad a_{02} = 4: \ \{1,-4,6\}, \qquad a_{03} = 6: \ \{-2,-5,6\}.$$

A suitable choice of gauge is

$$G = 0, \qquad H = 0.$$

The first set of resonances gives nine conditions for the absence of logarithms, leaving one arbitrary function $L(x)$. We rename this function as $3R(x)$. The full version of F-XV is found to be

$$\begin{aligned}\text{F-XV :} \qquad y^{(\mathrm{iv})} \;=\; & -2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y' \\ & + 3R(x)(y'' + yy') + R'(x)(4y' + y^2) \\ & + R'''(x) - 2R(x)R'(x). \end{aligned} \tag{7.115}$$

A first integral of F-XV is

$$u'' = \frac{(u')^2 - (K_1)^2}{2u} - \tfrac{5}{3}u^2 + \big(\tfrac{3}{2}y^2 - R\big)u, \tag{7.116}$$

where $u$ denotes $y' + y^2 - R$ and $K_1$ is a constant of integration. The general solution of F-XV is given by solving the Riccati equation,

$$y' = -y^2 + \frac{3w'}{w}y + R - \frac{3w''}{w}, \tag{7.117}$$

where $w(x)$ is the solution of the third-order linear equation,

$$w''' = Rw' + \tfrac{1}{3}K_1 w. \tag{7.118}$$

The relationship between F-XV and Chazy-VI is as follows. If $R(x)$ is an elliptic (or simpler) function satisfying

$$R'' = \tfrac{1}{2}R^2 + \beta, \tag{7.119}$$

where $\beta$ is a constant, then the Riccati equation (7.117) has the particular solution,

$$y = \frac{U'}{U}, \qquad U = 2ww'w'' - (w')^3 - Rw^2w' + \tfrac{1}{3}\big(R' - K_1\big)w^3. \tag{7.120}$$

This $y$ is the general solution of the Chazy-VI equation,

$$\text{Chazy-VI :} \quad y''' = -yy'' - 5(y')^2 - y^2y' + 3R(x)y' + R'(x)y - R''(x). \tag{7.121}$$

*7.8. Equation F-XVI, generalization of Ince-VI.*

The reduced equation F-XVI is the only true Painlevé-type equation that arose from the 202511 discrete solutions of the Diophantine equation (4.51). Only one other equation in this class, namely, (4.7), has integer resonances subject to the usual rules, but it fails a condition for the absence of logarithms. The corresponding full equation takes the form,

$$\begin{aligned} y^{(\mathrm{iv})} &= -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5 \\ &\quad + G(x,y,y',y'',y'''), \end{aligned} \tag{7.122}$$

and the resonance data are

$$a_{01} = 1: \;\; \{1,2,3\}, \qquad a_{02} = 2: \;\; \{1,2,-2\},$$
$$a_{03} = 3: \;\; \{1,-2,-3\}, \qquad a_{04} = 4: \;\; \{-2,-3,-4\}.$$

This is the easiest case to examine because the resonances in the first set are as small as possible. We place no restrictions on the gauge. The result is the following form for F-XVI:

$$\begin{aligned} \text{F-XVI}: \qquad y^{(\mathrm{iv})} &= -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5 \\ &\quad + A(x)\big(y''' + 4yy'' + 3(y')^2 + 6y^2y' + y^4\big) \\ &\quad + B(x)(y'' + 3yy' + y^3) \\ &\quad + C(x)(y' + y^2) + D(x)y + E(x). \end{aligned} \tag{7.123}$$

The coefficient functions are arbitrary. The solution of F-XVI is

$$y = u'/u, \tag{7.124}$$

where $u(x)$ satisfies the fifth-order linear equation,

$$u^{(\mathrm{v})} = A(x)u^{(\mathrm{iv})} + B(x)u''' + C(x)u'' + D(x)u' + E(x)u. \tag{7.125}$$

Equation F-XVI and its solution was found by both Exton [25] in his analysis of fourth-order equations. Martynov [21] obtained the reduced equation but did not proceed further.

Equation F-XVI generalizes trivially to all differential orders, like some of the other linearizable equations encountered above. The third-order and second-order counterparts are, respectively, Chazy-XII with $N = 2$ and Ince-VI in general gauges. More generally, equations F-VIII, F-IX, F-XII, F-XVI and higher-order generalizations thereof can all be constructed by nesting equation (7.124) with one or two linear differential equations. Similarly, equation F-XIII and higher-order generalizations thereof can be constructed by nesting Chazy-XII with $N = 3$ and a linear equation. Exton, in the conclusion to his paper on higher-order Painlevé classification [25], claimed that all equations in the polynomial class of orders six and above can be constructed in one or the other of these two ways. If that were true, there would be no hyperelliptic or Painlevé hierarchies.

### *7.9. Equation F-XVII, fourth-order transcendent.*

Equation F-XVII is one of the earliest differential equations found that defines a higher-order Painlevé transcendent, and perhaps the second that was found by direct Painlevé classification. It was found (minus a parameter) by Martynov in 1973 [21] and rediscovered independently by Ablowitz and Segur in 1977 [44] and Flaschka and Newell in 1980 [45]. It has come to be known as the second member of the Painlevé-II hierarchy [56, 57], although not all authors agree on what constitutes the most natural hierarchy built on Painlevé-II. It is related to equations F-V and Fif-III in paper I and can be obtained by self-similar reduction of the Modified KdV5 equation (4.5).

Earlier examples of higher-order Painlevé transcendent equations are Fif-I [1, (2.69)], discovered by Chazy [6, p. 345], and higher-order generalizations of Painlevé-VI, constructed by Garnier [28]. The work of Garnier has been generalized by the Kyoto school [58, 59, 60, 61, 62, 63]. Despite the fact that higher-order Painlevé transcendents have been well known since the 1970s, studies of their function-theoretic properties are still very rare [64, 65, 66].

In the case of F-XVII, the full equation to be determined takes the form,

$$y^{(\mathrm{iv})} \;=\; 10y^2y'' + 10y(y')^2 - 6y^5 + G(x, y, y', y'', y'''). \tag{7.126}$$

The resonance data are

$$a_{01} = 1: \;\; \{2, 3, 6\}, \qquad a_{02} = -1: \;\; \{2, 3, 6\},$$
$$a_{03} = 2: \;\; \{-3, 6, 8\}, \qquad a_{04} = -2: \;\; \{-3, 6, 8\}.$$

We examine the first two sets of resonances together. A choice of gauge that sets $a_1 = 0$ in both corresponding Laurent series is

$$J \;=\; -6G, \qquad \tilde{K} \;=\; -(2H + I).$$

The resonance 2 gives the constraints,

$$M \;=\; 0, \qquad N \;=\; -2L.$$

The resonance 3 gives the constraints,

$$G = 0, \qquad I = -H, \qquad P = 0, \qquad Q = 0.$$

Finally the resonance 6 gives the constraints,

$$H = 0, \qquad L' = 0, \qquad R'' = 0, \qquad S' = 0.$$

We conclude that the nonzero coefficients are

$$L = -10\delta, \qquad N = 20\delta, \qquad R = \lambda x + \mu, \qquad S = \sigma, \tag{7.127}$$

where $\delta$, $\lambda$, $\mu$, and $\sigma$ are constants. Hence the full form of equation F-XVII is

$$\text{F-XVII}: \qquad y^{(\mathrm{iv})} = 10y^2y'' + 10y(y')^2 - 6y^5 \tag{7.128}$$

$$- 10\delta\big(y'' - 2y^3\big) + (\lambda x + \mu)y + \sigma. \tag{7.129}$$

Martynov found this equation with the restriction $\delta = 0$. The version given by putting $n = 2$ in the hierarchy of Ablowitz and Segur [44] contains integrals. If these integrals are allowed to take arbitrary constants of integration, then the equation is seen to be complete after suitable rearrangement. Similarly, the version given by putting $n = 2$ in the hierarchy of Flaschka and Newell [45] is also complete.

We claim that, like F-V, F-VI, and F-XVIII, equation F-XVII is a differential equation that defines an irreducible fourth-order Painlevé transcendent when $\lambda \neq 0$. We will not try to prove this claim here. Proofs of this nature require the powerful machinery of differential Galois theory. Our argument in support of this hypothesis is that F-XVII has particular solutions involving the classical Painlevé-I transcendent and limiting and asymptotic solutions involving hyperelliptic functions of genus 2. The case $\lambda = 0$ can be solved completely with hyperelliptic functions. Hence the solution of the case $\lambda \neq 0$ is a one-parameter deformation of this hyperelliptic solution. We do not see a way to construct such a deformation out of classical transcendents, including any of the six Painlevé transcendents, abelian functions, Kleinian functions, etc., in such a way as to capture a particular solution involving the Painlevé-I transcendent.

We should also remark at this stage that the foregoing analysis does not prove that F-XVII (or, similarly, F-V, F-VI, F-XVIII) has the Painlevé property. We have only run tests that provide necessary conditions. Several techniques are available for completing these proofs, including the proofs of the stronger statements that the solutions of these equations do not admit any essential singularities except at infinity [67, 68, 69, 70].

Equation F-XVII admits a Schlesinger transformation that can be deduced from the Bäcklund transformation for the Modified KdV5 equation. Let $\epsilon_1 = \pm 1$. The transformation is

$$\bar{y} \,=\, -\,y - \frac{\epsilon_1 \sigma + \frac{1}{2}\lambda}{D}\,, \qquad \bar{\sigma} \,=\, \sigma + \epsilon_1 \lambda, \tag{7.130}$$

where

$$D \,:=\, y''' - 2(3y^2 - 5\delta)y' + \epsilon_1\big\{2yy'' - (y')^2 - 3y^4 + 10\delta y^2 + \tfrac{1}{2}(\lambda x + \mu)\big\}. \tag{7.131}$$

Equation F-XVII also admits the trivial involution, $y \to -y$, $\sigma \to -\sigma$.

The Painlevé-I solution arises because Chazy-XI with $N = 3$ is a particular first integral of F-XVII when $\sigma = \lambda/2$. With the sign of $y$ changed and parameters renamed in the standard form of Chazy-XI [19], we get the third-order equation,

$$y''' \,=\, 2yy'' + 2(y')^2 - 3\big(y' - y^2 + \tfrac{5}{3}\delta\big)^2 + \tfrac{1}{2}(\lambda x + \mu) + \tfrac{25}{3}\delta^2. \tag{7.132}$$

All solutions of (7.132) satisfy F-XVII with $\sigma = \lambda/2$. Equation (7.132) is solved by nesting a Riccati equation and a Painlevé-I equation as follows:

$$y' \,=\, y^2 - 2w - \tfrac{5}{3}\delta, \qquad w'' \,=\, 6w^2 - \tfrac{1}{4}(\lambda x + \mu) - \tfrac{25}{6}\delta^2. \tag{7.133}$$

When $\lambda = 0$, equation (7.132) can be solved in terms of elliptic functions and quadratures or simpler functions. Solutions involving Painlevé-I can be constructed for $\sigma = (n + 1/2)\lambda$, where $n$ is any integer.

The hyperelliptic solution is given by the following construction. Let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{dt}{\sqrt{P(t)}} \,=\, K_3\,, \tag{7.134}$$

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} \,=\, x + K_4\,, \tag{7.135}$$

where $P(t)$ is the quintic polynomial,

$$P(t) := t^5 - 2\alpha t^3 + 8\beta t^2 + 32K_1 t + 16K_2\,, \tag{7.136}$$

and

$$\alpha \,=\, \mu + 30\delta^2, \qquad K_2 \,=\, \tfrac{1}{4}\sigma^2 - 8\delta(K_1 - 22\delta^4 - \mu\delta^2 + \beta\delta). \tag{7.137}$$

The general solution of F-XVII with $\lambda = 0$ is

$$y = -\frac{1}{2}\left\{\frac{u'}{u-4\delta} + \frac{v'}{v-4\delta}\right\} + \frac{\sigma}{(u-4\delta)(v-4\delta)}. \tag{7.138}$$

The four constants of integration are $\beta$, $K_1$, $K_3$, and $K_4$. There are a number of special cases where this hyperelliptic solution degenerates into elliptic or simpler functions.

Equation F-XVII admits the rational solution,

$$y = D + \frac{3(Dx^4 + 2x^3 + 2CDx + C)}{(x^3 - C)(D^2x^3 + 3Dx^2 + 3x - CD^2)}, \tag{7.139}$$

when $\delta = D^2$, $\lambda = 0$, $\mu = -30D^4$, and $\sigma = 16D^5$.

We refer the reader to the Appendix of paper I for the Bäcklund transformation connecting F-XVII with the Fif-III equation,

$$\text{Fif-III}: \qquad y^{(\mathrm{v})} = 20yy''' + 40y'y'' - 120y^2y' + (\lambda x + \alpha)y' + 2\lambda y + \kappa, \tag{7.140}$$

and its first integral [1, (5.20)]. Because $\kappa = \lambda\delta$, the particular case of Fif-III with $\lambda = 0$ and $\kappa \neq 0$ does not participate in this direct relationship with F-XVII but can be attained via an obvious limit. It is this exceptional case that admits F-V as a first integral.

An alternative and more direct way to obtain F-V as a limit of F-XVII has been provided by Littlewood [71], who searched for a contraction analogous to the well-known contraction from Painlevé-II to Painlevé-I [27, 72]. Littlewood's contraction is

$$\tilde{x} = \epsilon^2 x - 30\epsilon^{-28}\kappa^{-1}, \qquad \tilde{y} = \epsilon^3 y + \epsilon^{-7},$$
$$\tilde{\delta} = \tfrac{1}{60}\epsilon^6\alpha + \epsilon^{-14}, \qquad \tilde{\lambda} = \kappa,$$
$$\tilde{\mu} = \epsilon^2\beta, \qquad \tilde{\sigma} = -\tfrac{1}{3}\epsilon^{-15} + 16\epsilon^{-35}.$$

If $\tilde{y}(\tilde{x})$ satisfies (7.129) with tildes on all variables and parameters, then, in the limit $\epsilon \to 0$, $y(x)$ satisfies (1.8). This limit lends support to the argument that F-XVII is the natural choice for the second member of the Painlevé-II hierarchy. Another argument is that the self-similar reduction of the Modified KdV equation is the Painlevé-II equation (after integrating Chazy-VIII), and so the same should be true for the corresponding hierarchies. However, some authors (e.g., Kitaev [65]), working from the point of view of isomonodromy theory, have argued in favor of a different hierarchy built upon Painlevé-II. In that case, the second member would be the equation,

$$
\begin{aligned}
y^{(\mathrm{iv})} &= \frac{1}{y}\Big\{2y'y''' + \tfrac{3}{2}(y'')^2 + 5y^3y'' + \tfrac{5}{2}y^2(y')^2 \\
&\qquad - \tfrac{5}{2}y^6 + \alpha y^4 - (\lambda x + \mu)(y'' - y^3) - \lambda y' \\
&\qquad + \beta y^2 + \tfrac{1}{8}(\lambda x + \mu)^2\Big\} - \frac{(y')^2}{y^2}\Big\{2y'' - \lambda x - \mu\Big\}, \qquad (7.141)
\end{aligned}
$$

which is a rational equation with the Bureau symbol $\{P1, Z2\}$.

*7.10. Equation F-XVIII, fourth-order transcendent.*

Equation F-XVIII also defines a fourth-order Painlevé transcendent. Like F-VI in paper I, it was not found in previous attempts at the Painlevé classification of fourth-order equations. It is, however, well known to be the self-similar reduction of the Modified Sawada-Kotera equation (4.4) found by Fordy and Gibbons [20]. The latter authors found the relationship between the Sawada-Kotera and Kaup-Kupershmidt equations by giving the Miura transformations from (4.4) to both equations. There is a corresponding relation connecting the fifth-order equations Fif-I and Fif-II [1, (2.69), (2,70)] with each other and with F-XVIII, except that the Riccati equations involved in the Miura transformations can be integrated explicitly (see paper I). The hierarchy of Painlevé equations built on F-XVIII has been studied recently by Joshi and Pickering [43].

The full equation to be determined takes the form,

$$
y^{(\mathrm{iv})} = -5y'y'' + 5y^2y'' + 5y(y')^2 - y^5 + G(x, y, y', y'', y'''). \qquad (7.142)
$$

The resonance data are

$$
\begin{aligned}
a_{01} &= 1: \ \{2, 3, 6\}, \qquad a_{02} = -2: \ \{2, 3, 6\}, \\
a_{03} &= -3: \ \{-2, 6, 7\}, \qquad a_{04} = 4: \ \{6, -7, 12\}.
\end{aligned}
$$

We examine the first two sets of resonances together. A choice of gauge that sets $a_1 = 0$ in both corresponding Laurent series is

$$
J = -\tfrac{1}{2}(9G - 2H - I), \qquad \tilde{K} = \tfrac{1}{2}(3G - 2H - I).
$$

The resonance 2 gives the constraints,

$$
M = L, \qquad N = -L.
$$

The resonance 3 gives the constraints,

$$H = -G, \qquad I = 3G, \qquad P = 0, \qquad Q = 0.$$

Finally the resonance 6 gives the constraints,

$$G = 0, \qquad L = 0, \qquad R'' = 0, \qquad S' = 0.$$

We conclude that the nonzero coefficients are

$$R = \lambda x + \alpha, \qquad S = \gamma, \tag{7.143}$$

where $\lambda$, $\alpha$, and $\gamma$ are constants. Hence the full form of equation F-XVIII is

$$\text{F-XVIII}: \quad y^{(\mathrm{iv})} = -5y'y'' + 5y^2y'' + 5y(y')^2 - y^5 + (\lambda x + \alpha)y + \gamma. \tag{7.144}$$

We conjecture that this equation defines a fourth-order Painlevé transcendent when $\lambda \neq 0$ and $\gamma \neq (3n - 1/2)\lambda$ where $n$ is an integer. As in the case of F-XVII, we base this conclusion on the fact that F-XVIII contains multi-parameter particular solutions involving Painlevé-I transcendents and hyperelliptic functions. There are not enough free parameters in F-XVIII to build a solution containing these limits out of classical transcendents.

We refer the reader to paper I for the mappings between F-XVIII and Fif-I and Fif-II. From these maps, we can deduce two involutions acting on F-XVIII. First,

$$\bar{y} = y + \frac{2(\lambda - \gamma)}{D_1}, \qquad \bar{\gamma} = 2\lambda - \gamma, \tag{7.145}$$

where

$$D_1 := y''' - yy'' + 3(y')^2 - 4y^2y' + y^4 - \lambda x - \alpha. \tag{7.146}$$

Secondly,

$$\bar{y} = y + \frac{2\gamma + \lambda}{D_2}, \qquad \bar{\gamma} = -(\gamma + \lambda), \tag{7.147}$$

where

$$D_2 := 2y''' + 4yy'' + 3(y')^2 - 3y^2y' - y^4 + \lambda x + \alpha. \tag{7.148}$$

The products of these involutions in either order give the Schlesinger transformations, which raise or lower the parameter $\gamma$ by integer multiples of $3\lambda$. The simplest case, where $\bar{\gamma} = \gamma + 3\lambda$, is

$$\bar{y} = y + \frac{2\gamma + \lambda}{D_2} + \frac{2(\gamma + 2\lambda)D_2}{D_1D_2 - 3(2\gamma + \lambda)(y'' + 2yy')}. \tag{7.149}$$

This can also be deduced from the Bäcklund transformations admitted by the Sawada-Kotera and Kaup-Kupershmidt equations [73, 74, 75].

When $\gamma = (3n - 1/2)\lambda$, equation F-XVIII can be solved in terms of two Painlevé-I functions. The general solution in the case $\gamma = -(1/2)\lambda$ is

$$y = \frac{u' - v'}{u - v}, \tag{7.150}$$

where $u(x)$ and $v(x)$ are any two solutions of the Painlevé-I equation,

$$u'' = 6u^2 - \tfrac{1}{24}(\lambda x + \alpha). \tag{7.151}$$

If just one of $u(x)$ and $v(x)$ is eliminated, $y$ satisfies the Ince-X equation (found by Painlevé [55]),

$$y'' = -yy' + y^3 - 12u(x)y + 12u'(x), \tag{7.152}$$

and the same equation with $v(x)$ replacing $u(x)$. The solution is expressible in terms of elliptic or simpler functions when $\lambda = 0 = \gamma$. Similarly, when $\gamma = (5/2)\lambda$, the general solution of F-XVIII is

$$\begin{aligned} y = {} & \frac{u' - v'}{u - v} \\ & - \frac{\frac{1}{6}\lambda(u - v)}{(u')^2 - (v')^2 - 4u^3 + 4v^3 + \frac{1}{12}(\lambda x + \alpha)(u - v)}. \end{aligned} \tag{7.153}$$

When $\lambda = 0$ and $\gamma \neq 0$, F-XVIII can be solved in terms of hyperelliptic functions of genus 2. Let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{dt}{\sqrt{P(t)}} = K_3\,, \tag{7.154}$$

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = x + K_4\,, \tag{7.155}$$

where $P(t)$ is the sextic polynomial,

$$P(t) := (t^2 + A)^3 - \tfrac{1}{3}\alpha(t^2 + A) - \tfrac{2}{9}\gamma t + \tfrac{1}{3}\beta. \tag{7.156}$$

Then the general solution of F-XVIII with $\lambda = 0$ and $\gamma \neq 0$ is

$$y = \frac{u' + v'}{u + v} + u + v. \tag{7.157}$$

The constants $A$, $\beta$, $K_3$, and $K_4$ are the four integration constants. These symbols are consistent with the usage in paper I for the related equation F-III. There are a number of cases where these hyperelliptic functions degenerate to elliptic or simpler functions.

The reduced F-XVIII equation admits the rational solution,

$$y = \frac{4x^7 + 3C}{x(x^7 - C)}, \tag{7.158}$$

where $C$ is a constant.

### *7.11. Equation F-XIX, Clarkson-Olver hierarchies.*

Equation F-XIX, which we call the Clarkson-Olver equation, is the differential equation,

$$\text{F-XIX}: \qquad \begin{aligned} y^{(\mathrm{iv})} &= \tfrac{10}{3} y y''' - 4y'y'' - \tfrac{8}{3} y^2 y'' + 4y(y')^2 \\ &\quad + 4\alpha\big(6y' - y^2\big)\big(3y'' - 3yy' + \tfrac{1}{3}y^3\big). \end{aligned} \tag{7.159}$$

It is not a Painlevé-type equation, but will be seen to be integrable in terms of Lamé functions in the same manner as Chazy-XII. It is the simplest generalization of Chazy-XII suggested by the group-theoretic methods of Clarkson and Olver [22, 76] and can be reduced to a first-order Abel equation by their methods. The solution given here is new. It was found with the aid of the auxiliary Abel equation, but is presented in a direct manner.

In [22], Clarkson and Olver presented three large classes of higher-order differential equations that can have their differential order reduced by three by group-theoretic methods generalizing those of Lie. If the lower-order equation is integrable, then the solution of the original equation can be recovered by a Riccati equation and quadratures. The question of when these equations are integrable is a potentially important one in the analytic theory of differential equations. Here, we consider the question very briefly and restrict attention to the third of the three Clarkson-Olver families of differential equations because this one contains equations in the polynomial class.

The differential equations in the third Clarkson-Olver family are built out of $\mathrm{SL}(2, \boldsymbol{C})$ differential invariants which are themselves ratios of relative invariants defined inductively by

$$J_1 = y' - \tfrac{1}{6}y^2, \qquad J_n = \left\{\frac{d}{dx} - \tfrac{1}{3}ny\right\} J_{n-1}. \tag{7.160}$$

By giving $J_n$ a homogeneity weight of $n+1$, the differential equations under consideration take the form of weighted homogeneous polynomials in the $J_i$ set equal to zero. (We have chosen a scaling factor so as to keep Chazy-XII in its standard form.) The first four relative invariants after $J_1$ and useful relatives thereof are

$$\begin{aligned} J_2 &= y'' - yy' + \tfrac{1}{9}y^3, \\ J_3 &= \tilde{J}_3 - 4(J_1)^2, \\ \tilde{J}_3 &:= y''' - 2yy'' + 3(y')^2, \\ J_4 &= \tilde{J}_4 - 8J_1J_2\,, \\ \tilde{J}_4 &:= y^{(\mathrm{iv})} - \tfrac{10}{3}yy''' + 4y'y'' + \tfrac{8}{3}y^2y'' - 4y(y')^2, \\ J_5 &= \tilde{J}_5 - \tfrac{172}{3}J_1\tilde{J}_3 + 46(J_2)^2 + 176(J_1)^3, \\ \tilde{J}_5 &:= y^{(\mathrm{v})} - 5yy^{(\mathrm{iv})} + 50y'y''' - 50(y'')^2. \end{aligned}$$

With this notation, the Chazy-III and Chazy-XII equations become, respectively,

$$\tilde{J}_3 = 0, \qquad \tilde{J}_3 - 36\alpha(J_1)^2 = 0, \tag{7.161}$$

where $\alpha = 4/(36-N^2)$. In the Clarkson-Olver theory, the parameter $\alpha$ is allowed to take arbitrary complex values.

Equation F-XIX takes the form,

$$\tilde{J}_4 - 72\alpha J_1J_2 = 0. \tag{7.162}$$

When $\alpha \neq 0$, its resonance data are

$$\begin{aligned} &a_{01} = -6: \ \big\{-2,-3,-4\big\}, \qquad a_{02} = -3: \ \big\{1,\, \delta\sqrt{3\alpha}\,, -\delta\sqrt{3\alpha}\,\big\}, \\ &a_{03} = \tfrac{1}{2}(\delta-6): \ \big\{1,\, \delta,\, \tfrac{2}{3}\delta\big\}, \qquad a_{04} = -\tfrac{1}{2}(\delta+6): \ \big\{1, -\delta, -\tfrac{2}{3}\delta\big\}, \end{aligned}$$

where $\delta := 2\sqrt{(9\alpha-1)/\alpha}$. When $\alpha = 0$, the resonance data are

$$a_{01} = -6: \ \big\{-2,-3,-4\big\}, \qquad a_{02} = -3: \ \big\{1,\, 2i\sqrt{3}\,,\, -2i\sqrt{3}\,\big\}.$$

Equation F-XIX has the Painlevé property only in the special case $\alpha = 4/27$, in which case F-XIX is equivalent under scaling to the reduced form of F-XVI, whose solution is a rational function.

In terms of the absolute invariants,

$$R := \tilde{J}_3/(J_1)^2, \qquad h := J_2/(J_1)^{3/2}, \tag{7.163}$$

equation F-XIX can be written as the first-order Abel equation [76],

$$\frac{dR}{dh} \;=\; -\,\frac{4h(R-36\alpha)}{2R-3h^2-8}\,. \tag{7.164}$$

Because this equation is gauge-equivalent to an Abel equation with constant coefficients, it can be readily integrated. The result is

$$\big(R-36\alpha\big)^3 \;=\; K\big(6R-3h^2-144\alpha-8\big)^2, \tag{7.165}$$

where $K$ is the constant of integration. In terms of the relative invariants, the latter equation reads

$$\big\{\tilde{J}_3-36\alpha(J_1)^2\big\}^3 \;=\; K\big\{6J_1\tilde{J}_3-3(J_2)^2-8(18\alpha+1)(J_1)^3\big\}^2, \tag{7.166}$$

which is a differential equation of the third order and third degree for $y$.

Equation (7.166) is a Clarkson-Olver equation of the third kind and can be integrated in terms of a Riccati equation and quadratures by the methods of [22]. (Actually, the quadratures can be bypassed by eliminating some of the auxiliary variables.) A naive application of the method produces a messy solution involving the roots of the cubic equation (7.165) for $R$. However, by a judicious choice of variables, the solution can be presented in a very neat form.

Let $\psi(t)$ and $\psi_1(t)$ be any two linearly independent solutions of the linear differential equation,

$$\begin{aligned}&\big\{(9\alpha-1)t^3+3Kt-1\big\}\ddot{\psi}\\ &\qquad\qquad+\tfrac{3}{2}\big\{(9\alpha-1)t^2+K\big\}\dot{\psi}\;-\;\tfrac{1}{16}t\psi\;=\;0,\end{aligned} \tag{7.167}$$

where the dot denotes differentiation with respect to $t$. Equation (7.167) is a Lamé differential equation in rational form. The general solution of F-XIX and (7.166) is given parametrically by

$$x \;=\; \frac{\psi_1}{\psi}\,, \qquad y \;=\; \frac{6\psi\dot{\psi}}{\psi\dot{\psi}_1-\psi_1\dot{\psi}}\,. \tag{7.168}$$

When $K=0$, this solution reduces to the general solution of the Chazy-XII equation, in which case the Lamé equation can be transformed into a hypergeometric equation. When $\alpha=1/9$, the Lamé equation simplifies and can be solved in terms of parabolic cylinder functions (Airy functions if $K=0$). Included in the subcase $K=0$ is the singular integral of equation (7.166). This integral, which makes both sides of (7.166) vanish separately, is just the well-known two-parameter rational solution of Chazy-XII.

There are advantages in using the Clarkson-Olver theory to lower the order of differentiation by one rather than three. We ignore the solvable subgroup of SL$(2, \boldsymbol{C})$ consisting of scalings and translations. Suppose $g(t)$ is a function of $t$ to be determined and $z(t)$ and $z_1(t)$ are linearly independent solutions of the linear differential equation,

$$\ddot{z} + g(t)z \;=\; 0. \tag{7.169}$$

If $y(x)$ satisfies a Clarkson-Olver equation of the third kind of order $n$, then the ansatz,

$$x \;=\; \frac{z_1}{z}, \qquad y \;=\; \frac{6z\dot{z}}{z\dot{z}_1 - z_1\dot{z}}, \tag{7.170}$$

yields a differential equation for $g(t)$ of order $n-1$.

When applied to F-XIX, this ansatz yields the reduced Chazy-XIII equation,

$$\dddot{g} + 48(9\alpha - 1)g\dot{g} \;=\; 0. \tag{7.171}$$

Two integrations yield

$$\dot{g}^2 \;=\; 16(1-9\alpha)g^3 + 2K_1 g + K_2\,, \tag{7.172}$$

where $K_1$ and $K_2$ are integration constants. The constant $K$ appearing above is related to these according to $K_1^3 + 54K^3K_2^2 = 0$. Hence, in general, $g(t)$ is an elliptic function and (7.169) is a Lamé equation in the form where an elliptic function appears as a potential.

If the Clarkson-Olver equation is a first-degree reduced equation in the polynomial class with Bureau symbol $P1$, then the $g(t)$-equation is a similar reduced equation having Bureau symbol $P2$. All such $g(t)$ equations of the fourth and fifth order having the Painlevé property are classified and solved in paper I. This yields immediately five one-parameter families of integrable Clarkson-Olver equations of order five and four of order six. (We are not including the Bureau barrier equation F-II in this discussion.) The parameter arises from the scaling freedom in the $g(t)$ equation but is a nontrivial parameter in the Clarkson-Olver equation. The linear equation $d^n g/dt^n = 0$ also generates an integrable Clarkson-Olver equation of order $n+1$. The known hyperelliptic hierarchies (e.g., the Drach hierarchy [13]) will generate corresponding hierarchies of integrable Clarkson-Olver equations. In general, the latter do not have the Painlevé property and have quite complicated singularity structures.

The fifth-order Clarkson-Olver equations in the polynomial class take the form,

$$J_5 + AJ_1J_3 + B(J_2)^2 + C(J_1)^3 \;=\; 0, \tag{7.173}$$

where $A$, $B$, and $C$ are complex constants. The sixth-order equations are obtained from these by applying the differential operator $d/dx - 2y$ to the left-hand side, the result being

$$J_6 + AJ_1J_4 + (A+2B)J_2J_3 + 3C(J_1)^2J_2 \;=\; 0. \tag{7.174}$$

In the fifth-order case, the corresponding $g(t)$ equation is

$$\frac{d^4g}{dt^4} \;=\; 6\big\{Ag\ddot{g} + B\dot{g}^2 - 6Cg^3\big\}, \tag{7.175}$$

and, in the sixth-order case, it is

$$\frac{d^5g}{dt^5} \;=\; 6\big\{Ag\dddot{g} + (A+2B)\dot{g}\ddot{g} - 18Cg^2\dot{g}\big\}, \tag{7.176}$$

which is just the derivative of (7.175).

Equation (7.175) has the Painlevé property in the six cases,

$$\begin{aligned}\big\{A,\,B,\,C\big\} \;=\; \big\{0,\,0,\,0\big\},\quad \big\{2\lambda,\,2\lambda,\,0\big\},\quad \big\{\tfrac{5}{2}\lambda,\,\tfrac{15}{8}\lambda,\,\tfrac{5}{12}\lambda^2\big\},&\\ \big\{5\lambda,\,0,\,\tfrac{5}{3}\lambda^2\big\},\quad \big\{\tfrac{10}{3}\lambda,\,\tfrac{5}{3}\lambda,\,\tfrac{10}{9}\lambda^2\big\},\quad \big\{3\lambda,\,\tfrac{3}{2}\lambda,\,\tfrac{2}{3}\lambda^2\big\},&\end{aligned} \tag{7.177}$$

which correspond to the equations $d^4g/dt^4 = 0$, F-I, F-III, F-IV, F-V, and F-VI, respectively. The functions $g(t)$ are, respectively, a polynomial, a Painlevé-I transcendent, a hyperelliptic function (three consecutive cases), and a combination of two elliptic functions. It follows that the fifth-order equation (7.173) is integrable in all these cases. The parameter $\lambda$ is an essential parameter in (7.173) and (7.174). The sixth-order equation (7.174) is integrable for the same values of $\{A, B, C\}$ except for the case $\{2\lambda, 2\lambda, 0\}$. Here $g(t)$ is a polynomial in the first case and is a hyperelliptic function in the remaining four cases. Full details of the construction of these hyperelliptic functions in terms of standard hyperelliptic functions are given in paper I.

The Clarkson-Olver equation involving the Painlevé-I transcendent is especially interesting. It is a direct generalization of F-XIX and Chazy-XII and is the first example of which we are aware where an integrable equation contains both Chazy-XII and a Painlevé transcendent. Its singularity structure is very complicated. With $\lambda = 4(1-9\alpha)$, the equation can be put in the form,

$$\left(\frac{d}{dx} - \tfrac{5}{3}y\right)\left(\frac{d}{dx} - \tfrac{4}{3}y\right)\Big\{y''' - 2yy'' + 3(y')^2 - \alpha\big(6y' - y^2\big)^2\Big\} \;=\; 0, \tag{7.178}$$

Its solution is given by (7.169) and (7.170) above with $g(t)$ satisfying the Painlevé-I equation,

$$\ddot{g} \;=\; 24(1-9\alpha)g^2 + K_1t + K_2\,. \tag{7.179}$$

Another set of integrable fifth- and sixth-order Clarkson-Olver equations arises by Painlevé classifying the classes (7.173) and (7.174) directly. The details of such calculations will be postponed to a future paper. The provisional results are that (7.173) has the Painlevé property (or at least passes a significant number of necessary conditions for that property) in the twelve cases where $\{A, B, C\}$ takes the values,

$$\left\{\tfrac{172}{3},\, -46,\, \tfrac{160}{3}\right\}, \quad \left\{\tfrac{118}{3},\, -\tfrac{29}{2},\, \tfrac{196}{3}\right\}, \quad \left\{\tfrac{97}{3},\, \tfrac{37}{3},\, \tfrac{980}{9}\right\},$$
$$\left\{\tfrac{122}{3},\, -\tfrac{1}{6},\, \tfrac{980}{9}\right\}, \quad \left\{-\tfrac{31}{6},\, -\tfrac{13}{3},\, \tfrac{5}{4}\right\}, \quad \left\{-\tfrac{134}{9},\, -\tfrac{253}{18},\, \tfrac{320}{81}\right\},$$
$$\left\{-\tfrac{26}{3},\, -\tfrac{11}{2},\, \tfrac{16}{3}\right\}, \quad \left\{-26,\, -\tfrac{281}{36},\, \tfrac{160}{3}\right\}, \quad \left\{\tfrac{898}{21},\, \tfrac{31}{14},\, \tfrac{18496}{147}\right\},$$
$$\left\{\tfrac{757}{18},\, \tfrac{11}{9},\, \tfrac{41405}{324}\right\}, \quad \left\{\tfrac{299}{6},\, -19,\, \tfrac{1225}{12}\right\}, \quad \left\{\tfrac{463}{12},\, -\tfrac{11}{2},\, \tfrac{4225}{48}\right\}.$$

Two more cases, namely,

$$\left\{\tfrac{1336}{39},\, \tfrac{56}{39},\, \tfrac{10240}{117}\right\}, \quad \left\{\tfrac{664}{21},\, \tfrac{128}{7},\, \tfrac{2560}{21}\right\},$$

admit integer resonances but fail a test for the absence of logarithms. Only one member of this list, namely, $\{-26/3, -11/2, 16/3\}$, yields a sixth-order equation with the Painlevé property. We conjecture that there are an increasing number of Painlevé-type Clarkson-Olver equations at each odd differential order.

An important task for the future is to decide the integrability of these equations and find their solutions. A few have easily found rational solutions. The first member of this list is the interesting Fordy-Pickering equation [42],

$$y^{(\mathrm{v})} = 5yy^{(\mathrm{iv})} - 50y'y''' + 50(y'')^2, \tag{7.180}$$

which has just one set of resonances $\{-2, -3, -4, -5\}$ and is a direct generalization of Chazy-III. Every member that is found to be integrable implies a corresponding integrable, but non-Painlevé, equation of the form (7.175). Since the latter equation can be rescaled, the original Clarkson-Olver equation remains integrable, but loses the Painlevé property, after the rescaling,

$$\{A,\, B,\, C\} \to \{\lambda A,\, \lambda B,\, \lambda^2 C\}. \tag{7.181}$$

The variables $x$ and $y$ transform in a complicated manner under this mapping. Among the $g(t)$ equations appearing here is the equation

$$\frac{d^4 g}{dt^4} = \tfrac{13}{4} g\ddot{g} + \tfrac{33}{16}\dot{g}^2 - \tfrac{3}{4}g^3 + \text{const.}, \tag{7.182}$$

which was first formed by Bureau [50, p. 260] and solved in paper I.

It is clear that at least two distinct hierarchies are built on the Chazy-XII equation, one where the $y(x)$ equations have the Painlevé property together with their images under maps like (7.181), and the other where the $g(t)$ equations have the Painlevé property. The latter hierarchy is known to admit equations at all differential orders and includes complete hierarchies of Painlevé transcendents. The member (7.178) above involves the Painlevé-I transcendent. In a similar fashion, seventh-order Clarkson-Olver equations can be constructed that are integrable in terms of the full F-V and F-VI transcendents. We have not proved that the former hierarchy continues above the sixth order, but it appears likely that it admits increasing numbers of equations at all odd orders of differentiation. This will be a very interesting problem for further study.

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