The compact closed bicategory of left adjoints

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Abstract

We show that, for any braided compact closed bicategory $\mathcal{B}$, the bicategory $\mathbf{Ladj}(\mathcal{B})$ of left adjoints in $\mathcal{B}$ also admits a braided compact closed structure.

1 Introduction

Over the last few years monoidal bicategories have been studied in connection with physics and geometry ([BN], [C], [FY], [JS], [KV]) and of particular interest to these applications have been the notions of a braiding for a tensor and a dual for an object. In this paper we prove a theorem which deepens the connection between duals and adjoints, and which demonstrates how an old construction (the bicategory of squares [G]) can be used to define new braided compact closed bicategories.

Specifically, the main theorem of this paper states the following. Suppose $\mathbf{Sq}(\mathcal{B})$ is the monoidal bicategory of squares in a monoidal bicategory $\mathcal{B}$. Then an object $f : A \to B$ of $\mathbf{Sq}(\mathcal{B})$ has a right bidual if and only if (i) both A and B have right biduals in $\mathcal{B}$ and (ii) the arrow f has a right adjoint in $\mathcal{B}$.

A compact closed bicategory is defined to be a braided monoidal bicategory in which every object has a right bidual. A consequence of the above mentioned theorem is that the full sub-bicategory $\mathbf{Ladj}(\mathcal{B})$ of $\mathbf{Sq}(\mathcal{B})$ comprising left adjoints is compact closed if $\mathcal{B}$ is compact closed.

Instances of the construction of a bicategory $\mathbf{Ladj}(\mathcal{B})$ of left adjoints can be found in [C]. There, the notion of a *cobordism category* was defined

in order to give detailed descriptions of some of the categorical structures which arise in the study of topological quantum field theories. (In this paper we do not take into account the 'vertical' composition of squares that is considered in [C].) In the context of cobordism categories, typical candidates for $\mathcal{B}$ are the bicategories Cobord of cobordisms of compact, oriented smooth manifolds and the category Hilb of finite dimensional Hilbert spaces.

The original motivation for this work arose out of a general program to use monoidal bicategories with feedback for studying concurrency. The reader is referred to [KSW] for an overview of this work.

Most of the technical definitions in this paper are taken from [DS], and as is done there, we first deal with Gray monoids (which are strict monoidal bicategories), using the coherence theorem of [GPS] to transfer our defintions and results.

2 Preliminaries

2.1 Gray monoids

To begin with, we recall the data for a *Gray monoid* $\mathcal{M}$ from [DS]. It is a 2-category equipped with the following:

(a) an object I;

(b) for all objects A, two 2-functors $L_A, R_A : \mathcal{M} \to \mathcal{M}$ satisfying conditions

$$L_A(B) = R_B(A) \qquad \text{(and define } A \otimes B = L_A(B)\text{)},$$
$$L_I = R_I = 1_{\mathcal{M}},$$
$$L_{A\otimes B} = L_A L_B, \qquad R_{A\otimes B} = L_B L_A, \qquad R_B L_A = L_A R_B,$$

for all objects A, B; and

(c) for all arrows $f : A \to A'$, $g : B \to B'$, an invertible 2-cell

$$\begin{array}{ccc} A \otimes B & \xrightarrow{R_B(f)} & A' \otimes B \\ {\scriptstyle L_A(g)}\downarrow & \Downarrow c_{f,g} & \downarrow{\scriptstyle L_{A'}(g)} \\ A \otimes B' & \xrightarrow[R_{B'}(f)]{} & A' \otimes B' \end{array}$$

The structure 2-cells $c_{f,g}$ are subject to four axioms which we will not repeat here. Perhaps not apparent from these axioms is the fact that if either f or

g is an identity arrow then $c_{f,g}$ is an identity 2-cell. Henceforth, the symbol $\mathcal{M}$ will denote a Gray monoid.

In order to facilitate calculations in Gray monoids – in particular, the passage between algebraic and diagramatic representations of expressions – horizontal and vertical composition will be written diagramatically. If $f : A \to B$ and $f' : B \to C$ are arrows then $f \cdot f' : A \to C$ denotes their composite (we also use the symbol $\cdot$ to denote horizontal composition of 2-cells); if $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ are 2-cells then $\alpha \bullet \beta : f \Rightarrow h$ denotes their vertical composite.

Also, we put $L_A(g) = A \otimes g$, $L_A(\beta) = A \otimes \beta$, $R_B(f) = f \otimes B$, $R_B(\alpha) = \alpha \otimes B$, $L_A(g) \cdot R_{B'}(f) = f \otimes g$, and $L_A(\beta) \cdot R_{B'}(\alpha) = \alpha \otimes \beta$. In order to avoid the proliferation of symbols, we will sometimes suppress the symbol $\otimes$, for example we may write fg for $f \otimes g$; and, when the context is clear, we will omit the subcripts of the structure 2-cells $c_{f,g}$.

As in [DS], we write $\mathcal{M}^{\mathrm{op}}$ for the 2-category obtained from $\mathcal{M}$ by reversing the direction of arrows, and we write $\mathcal{M}^{\mathrm{co}}$ for the 2-category obtained from $\mathcal{M}$ by reversing the direction of 2-cells. If $\mathcal{M}$ is a Gray monoid then so are $\mathcal{M}^{\mathrm{op}}$ and $\mathcal{M}^{\mathrm{co}}$ in natural ways. Also, given $\mathcal{M}$, we can define another Gray monoid $\mathcal{M}^{\mathrm{rev}}$ with data $L'_A = R_A$, $R'_B = L_B$ and $c'_{f,g} = c^{-1}_{g,f}$; so we have, $A \otimes' B = B \otimes A$.

We refer to the objects, arrows and 2-cells of $\mathcal{M}^{\mathrm{co}}$, $\mathcal{M}^{\mathrm{op}}$ and $\mathcal{M}^{\mathrm{rev}}$ by the same names as their counterparts in $\mathcal{M}$. So, for example, if $s : XY \to Z$ is an arrow in $\mathcal{M}$, then there is an arrow $s : Z \to Y \otimes' X$ in $\mathcal{M}^{\mathrm{cooprev}}$.

2.2 Biduals

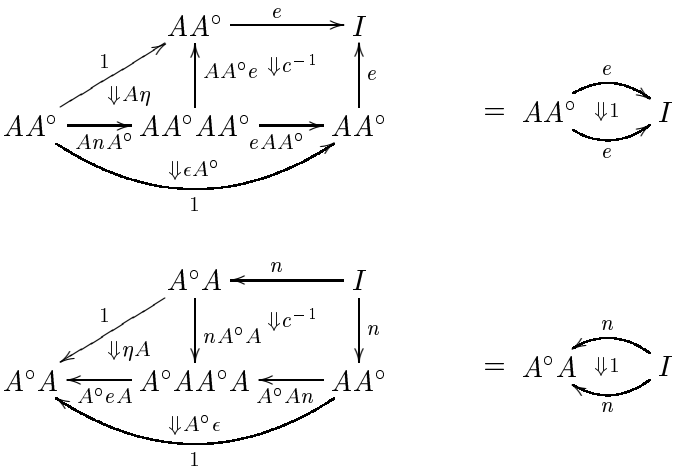
Recall that a *right bidual* for an object A of $\mathcal{M}$ comprises an object B, an arrow $e : A \otimes B \to I$ and, for all objects C, D, adjoint pseudo-inverses to the functor

$$e^{\#} : \mathcal{M}(C, B \otimes D) \to \mathcal{M}(A \otimes C, D),$$

given by $e^{\#}(f) = (A \otimes f) \cdot (e \otimes D)$. In such a case, the object B is often referred to as a right bidual for A (and A is called a left bidual for B). As right biduals are determined up to equivalence, we write A° for B. The following proposition [M] gives an alternative definition for biduals. Its proof is relatively straightforward, but involves non-trivial manipulations of the structure 2-cells c of $\mathcal{M}$.

Proposition 1 *To give a right bidual for A is to give an object A°, two arrows $e : A \otimes A^{\circ} \to I$ and $n : I \to A^{\circ} \otimes A$ as well as two invertible 2-cells*

$\epsilon : (A \otimes n) \cdot (e \otimes A) \Rightarrow 1_A$ *and* $\eta : 1_{A^\circ} \Rightarrow (n \otimes A^\circ) \cdot (A^\circ \otimes e)$ *such that the following two equations hold.*



If A has a right bidual as above we call $e : A \otimes A^\circ \to I$ a pairing and $n : I \to A^\circ \otimes A$ a copairing for A. Also, ϵ and η are respectively called the counit and unit isomorphisms for A. Of course, when we say $e : A \otimes A^\circ \to I$ is a pairing for A we mean there exists a bidual for A of the form $(A^\circ, e, n, \epsilon, \eta)$ (this quintuple being interpreted in the sense of the above proposition).

Notice that the following four conditions are equivalent.

(i) The arrow $f : X \otimes Y \to I$ is a pairing for X in $\mathcal{M}$.

(ii) The arrow $f : X \otimes Y \to I$ is a pairing for X in $\mathcal{M}^{\mathrm{co}}$.

(iii) The arrow $f : I \to X \otimes Y$ is a copairing for Y in $\mathcal{M}^{\mathrm{op}}$.

(iv) The arrow $f : Y \otimes' X \to I$ is a pairing for Y in $\mathcal{M}^{\mathrm{rev}}$.

2.3 Squares in $\mathcal{M}$

Let **2** be the category comprising the 'free living arrow'. For any 2-category **C**, define the 2-category $\mathbf{Sq}(\mathbf{C})$ of *squares* in **C** to be the 2-category of 2-functors, op-lax transformations and modifications from **2** to **C**. So an object of $\mathbf{Sq}(\mathbf{C})$ is an arrow $f : A \to A'$ in **C**, and an arrow from $f : A \to A'$ to $g : B \to B'$ in $\mathbf{Sq}(\mathbf{C})$ is a pair of arrows $h : A \to B$, $i : A' \to B'$ in **C**

together with a 2-cell of the form

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle g} \\ A' & \xrightarrow[i]{} & B' \end{array}$$

The arrows of $\mathbf{Sq}(\mathbf{C})$ are called squares in $\mathbf{C}$. There is an obvious pair of 2-functors $\mathrm{Dom},\ \mathrm{Cod} : \mathbf{Sq}(\mathbf{C}) \to \mathbf{C}$ which are locally fully faithful.

Proposition 2 *If $\mathcal{M}$ is a Gray monoid then $\mathbf{Sq}(\mathcal{M})$ naturally admits a Gray monoid structure such that the two 2-functors* $\mathrm{Dom},\ \mathrm{Cod} : \mathbf{Sq}(\mathcal{M}) \to \mathcal{M}$ *preserve the Gray monoid structure on the nose.*

Proof. The unit for the Gray monoid structure of $\mathbf{Sq}(\mathcal{M})$ is the object $1_I : I \to I$. Consider the following two squares.

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[l]{} & C' \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{k} & D \\ {\scriptstyle g}\downarrow & \Downarrow\beta & \downarrow{\scriptstyle i} \\ B' & \xrightarrow[m]{} & D' \end{array}$$

We define $L_f, R_i : \mathbf{Sq}(\mathcal{M}) \to \mathbf{Sq}(\mathcal{M})$ on objects by $L_f(i) = R_i(f) = f \otimes i$ and on arrows by:

$$L_f(\beta) = \begin{array}{ccc} AB & \xrightarrow{Ak} & AD \\ {\scriptstyle Ag}\downarrow & \Downarrow A\beta & \downarrow{\scriptstyle Ai} \\ AB' & \xrightarrow[Am]{} & AD' \\ {\scriptstyle fB'}\downarrow & \Downarrow c^{-1} & \downarrow{\scriptstyle fD'} \\ A'B' & \xrightarrow[A'm]{} & A'D' \end{array}, \qquad R_i(\alpha) = \begin{array}{ccc} AD & \xrightarrow{jD} & CD \\ {\scriptstyle Ai}\downarrow & \Downarrow c & \downarrow{\scriptstyle Ci} \\ AD' & \xrightarrow[jD']{} & CD' \\ {\scriptstyle fD'}\downarrow & \Downarrow \alpha D' & \downarrow{\scriptstyle hD'} \\ A'D' & \xrightarrow[lD']{} & C'D' \end{array}$$

There is a unique way to define L_f and R_i on 2-cells so that Dom and Cod preserve this structure. Similarly there is a unique choice for the structure 2-cells c of $\mathbf{Sq}(\mathcal{M})$. That this data is well defined is a straightforward exercise in the calculus of the structure 2-cells of a Gray monoid. That the structure 2-cells for $\mathbf{Sq}(\mathcal{M})$ satisfy the axioms for a Gray monoid is guaranteed by the fact that Dom and Cod are locally fully faithful. □

For any 2-category $\mathbf{C}$, there is an obvious isomorphism of 2-categories $\Gamma : \mathbf{Sq}(\mathbf{C})^{\mathrm{coop}} \to \mathbf{Sq}(\mathbf{C}^{\mathrm{coop}})$ which makes the following two triangles commute:

$$\begin{array}{ccc} \mathbf{Sq}(\mathcal{M})^{\mathrm{coop}} & \xrightarrow{\mathrm{Dom}^{\mathrm{coop}}} & \mathcal{M}^{\mathrm{coop}} \\ {\scriptstyle \mathrm{Cod}^{\mathrm{coop}}}\downarrow & \searrow^{\Gamma} & \uparrow{\scriptstyle \mathrm{Cod}} \\ \mathcal{M}^{\mathrm{coop}} & \xleftarrow[\mathrm{Dom}]{} & \mathbf{Sq}(\mathcal{M}^{\mathrm{coop}}) \end{array}$$

Though the 2-functor Γ only preserves the Gray monoid structure up to isomorphism, we have the following result.

Proposition 3 *For any Gray monoid, the 2-functor* $\Gamma : \mathbf{Sq}(\mathbf{C})^{\mathrm{cooprev}} \to \mathbf{Sq}(\mathbf{C}^{\mathrm{cooprev}})$ *preserves the Gray monoid structure on the nose.*

3 The main theorem

In this section we prove the following theorem.

Theorem 1 *Let* $f : A \to B$ *be an arrow of a Gray monoid* $\mathcal{M}$*. The object* f *of* $\mathbf{Sq}(\mathcal{M})$ *has a right bidual if and only if (i) both* A *and* B *have right biduals in* $\mathcal{M}$ *and (ii) the arrow* f *has a right adjoint in* $\mathcal{M}$*.*

The proof is divided into four parts.

3.1 Part 1

First we establish some notation. Suppose that X and Y are objects of $\mathcal{M}$ with right biduals X° and Y°. Let $(-)^\circ : \mathcal{M}(Y, X) \to \mathcal{M}(X^\circ, Y^\circ)$ be the functor mapping $j : Y \to X$ to

$$j^\circ = (n_Y X^\circ) \cdot (Y^\circ j X^\circ) \cdot (Y^\circ e_X) : X^\circ \to Y^\circ.$$

We call j° the opposite of j. We also define a functor $(-)^* : \mathcal{M}(X^\circ, Y^\circ) \to \mathcal{M}(Y, X)$ as follows: given an arrow $k : X^\circ \to Y^\circ$, define the arrow $k^* : Y \to X$ to be the composite $(Y n_X) \cdot (Y k X) \cdot (e_Y X)$.

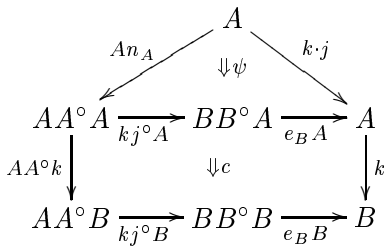
Let $\phi : (j^\circ)^* \Rightarrow j$ be the invertible 2-cell

$$[(Y n_Y n_X) \cdot c^{-1}] \bullet c^{-1} \bullet [\epsilon_Y \cdot j \cdot \epsilon_X].$$

Let $\psi : k \cdot j \Rightarrow (X n_X) \cdot (k j^\circ X) \cdot (e_Y X)$ be the invertible 2-cell defined by

$$[k \cdot \phi] \bullet [c \cdot (e_Y X)].$$

And let $\tau : k \cdot j \cdot k \Rightarrow (Xn_X) \cdot (kj^\circ k) \cdot (e_Y Y)$ be the invertible 2-cell



We call the 2-cell τ the torturing 2-cell of k and j.

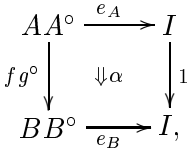
Recall that if $e_X : XX^\circ \to I$ is a pairing and $n_X : I \to X^\circ X$ is a copairing for X in $\mathcal{M}$ then $n_X : X \otimes' X^\circ \to I$ is a pairing and $e_X : X^\circ \otimes' X \to I$ is a copairing for X in $\mathcal{M}^{\text{cooprev}}$. The counit isomorphism for X in $\mathcal{M}^{\text{cooprev}}$ is ϵ_X^{-1}. Given an arrow $j : Y \to X$ in $\mathcal{M}$, the opposite of the arrow $j : X \to Y$ in $\mathcal{M}^{\text{cooprev}}$ is j° (that is, as an arrow in $\mathcal{M}$ it is the opposite of j). If $k : Y \to X$ is also an arrow in $\mathcal{M}$ then the torturing 2-cell of $k : X \to Y$ and $j : Y \to X$ in $\mathcal{M}^{\text{cooprev}}$ is τ^{-1}.

For the rest of this section, suppose that A and B are objects of $\mathcal{M}$ with right biduals A° and B° and that $f : A \to B$ and $g : B \to A$ are arrows.

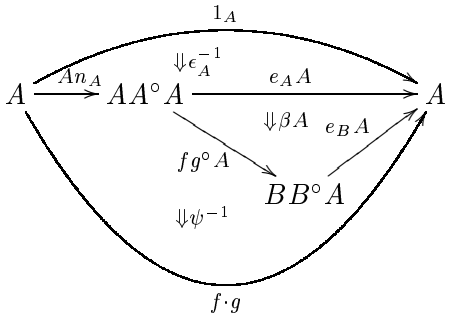
3.2 Part 2

At the heart of the proof are the following two constructions.

Construction 1 Given a square of the form



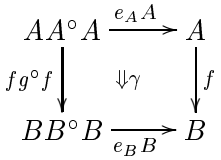
define the 2-cell $\tilde{\alpha} : 1_A \Rightarrow f \cdot g$ to be



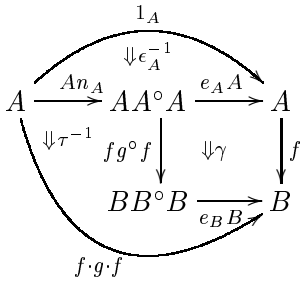
Proposition 4 *The above association $\alpha \mapsto \tilde{\alpha}$ establishes a bijection*

$$\mathcal{M}[AA^\circ, I](e_A, (fg^\circ) \cdot e_B) \cong \mathcal{M}[A, A](1_A, f \cdot g).$$

Construction 2 Given a square of the form



define the 2-cell $\overline{\gamma} : f \Rightarrow f \cdot g \cdot f$ to be



Given a square ι, define $\iota \otimes f$ to be the square $R_f(\iota)$, where R_f is part of the Gray monoid structure for $\mathbf{Sq}(\mathcal{M})$. Notice that if α is a square of the type mentioned in Construction 1 then $R_f(\alpha)$ is a square of the same type as γ mentioned above.

Proposition 5 *For any square α as above, we have the equality*

$$\overline{\alpha \otimes f} = \tilde{\alpha} \cdot f.$$

Proof. Using the third axiom (as given in [DS]) governing the structure 2-cells c of $\mathcal{M}$, we have $\alpha \otimes f = ((\alpha A) \cdot f) \bullet c$. A straightforward calculation yields the desired result. □

3.3 Part 3

Recall that if $n_X : I \to X^\circ X$ is a copairing for X in $\mathcal{M}$ then $n_X : X \otimes' X^\circ \to I$ is a pairing for X in $\mathcal{M}^{\text{cooprev}}$.

Construction 3 Consider a square in $\mathcal{M}$ of the form

$$\begin{array}{ccc} I & \xrightarrow{n_A} & A^\circ A \\ {\scriptstyle 1}\downarrow & \Downarrow\beta & \downarrow{\scriptstyle g^\circ f} \\ I & \xrightarrow[n_B]{} & B^\circ B \end{array}$$

Via $\Gamma : \mathbf{Sq}(\mathbf{C})^{\text{cooprev}} \to \mathbf{Sq}(\mathbf{C}^{\text{cooprev}})$, the corresponding square $\Gamma(\beta)$ in $\mathcal{M}^{\text{cooprev}}$ is of the form

$$\begin{array}{ccc} B \otimes' B^\circ & \xrightarrow{n_B} & I \\ {\scriptstyle f \otimes' g^\circ}\downarrow & \Downarrow\beta & \downarrow{\scriptstyle 1} \\ A \otimes' A^\circ & \xrightarrow[n_A]{} & I \end{array}$$

Notice that $\Gamma(\beta)$ is of the same form as the square α described in Construction 1. Thus we obtain a 2-cell $\tilde{\Gamma(\beta)} : 1_B \Rightarrow f \cdot g$ in $\mathcal{M}^{\text{cooprev}}$. Let us denote the corresponding 2-cell in $\mathcal{M}$ from $g \cdot f$ to 1_B by $\vec{\beta}$.

An immediate consequence of Proposition 4 is the following.

Proposition 6 *The above association $\beta \mapsto \vec{\beta}$ yields a bijection*

$$\mathcal{M}[I, B^\circ B](n_A \cdot (g^\circ f), n_B) \cong \mathcal{M}[B, B](g \cdot f, 1_B).$$

Construction 4 Consider a square in $\mathcal{M}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{An_A} & AA^\circ A \\ {\scriptstyle f}\downarrow & \Downarrow\delta & \downarrow{\scriptstyle fg^\circ f} \\ B & \xrightarrow[Bn_B]{} & BB^\circ B \end{array}$$

The square $\Gamma(\delta)$ is

$$\begin{array}{ccc} B \otimes' B^{\circ} \otimes' B & \xrightarrow{n_B \otimes' B} & B \\ {\scriptstyle f \otimes' g^{\circ} \otimes' f}\downarrow & \Downarrow \delta & \downarrow{\scriptstyle f} \\ A \otimes' A^{\circ} \otimes' A & \xrightarrow[n_A \otimes' A]{} & A \end{array}$$

This is of the same form as the square γ mentioned in Construction 2. So we obtain the 2-cell $\overline{\Gamma(\delta)} : f \Rightarrow f \cdot g \cdot f$ in $\mathcal{M}^{\mathrm{cooprev}}$. Refer to the corresponding 2-cell from $f \cdot g \cdot f$ to f in $\mathcal{M}$ as $\ddot{\overline{\delta}}$.

Let $L_f : \mathbf{Sq}(\mathcal{M}) \to \mathbf{Sq}(\mathcal{M})$ be the obvious 2-functor which is part of the Gray monoid structure on $\mathbf{Sq}(\mathcal{M})$. Write $f \otimes \beta$ for $L_f(\beta)$. Notice that if β is a square of the type mentioned in Construction 3 then $f \otimes \beta$ has the same form as the square δ described in Construction 4.

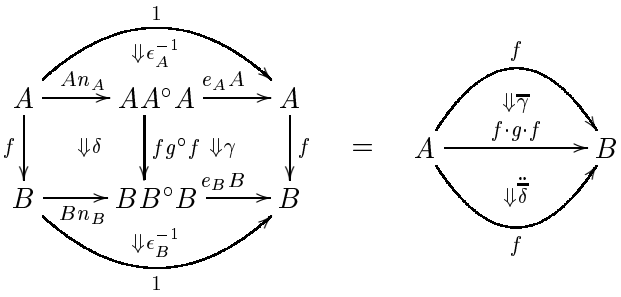
Let $R'_f : \mathbf{Sq}(\mathcal{M}^{\mathrm{cooprev}}) \to \mathbf{Sq}(\mathcal{M}^{\mathrm{cooprev}})$ be the obvious 2-functor which is part of the Gray monoid structure on $\mathbf{Sq}(\mathcal{M}^{\mathrm{cooprev}})$. Then for any square ι, it is straightforward to check that $\Gamma(f \otimes \iota) = R'_f(\Gamma(\iota))$. Combining this with Proposition 5, we obtain the following.

Proposition 7 *For any square β as above, we have the equality*

$$\ddot{\overline{f \otimes \beta}} = f \cdot \vec{\beta}.$$

Using the observations made in the penultimate paragragh of Part 1, we can evaluate the 2-cell $\ddot{\overline{\delta}}$ described in Construction 4. The following proposition is then immediate.

Proposition 8 *Given squares γ and δ as described in Constructions 2 and 4, the following equality holds.*



3.4 Part 4

In order to summarize the results of the preceeding parts, we introduce the following definitions. A right *proto-adjoint* for $f : A \to B$ is a triple (g, λ, μ) comprising an arrow $g : B \to A$ and two 2-cells $\lambda : 1_A \Rightarrow f \cdot g$ and $\mu : g \cdot f \Rightarrow 1_B$ such that $1_f = (\lambda \cdot f) \bullet (f \cdot \mu)$.

A right *proto-bidual* for X is a quadruple (Y, e, n, ϵ) comprising an object Y, arrows $e : XY \to I$ and $n : I \to YX$ and an isomorphism 2-cell $\epsilon : (Xn_X) \cdot (e_X X) \Rightarrow 1_X$.

When we say g is a right proto-adjoint for f, we mean that there exists a right proto-adjoint for f of the form (g, λ, μ). A similar interpretation is given to the phrase Y is a right proto-bidual for X.

Combining the results obtained so far, we get the following.

Proposition 9 *Suppose A and B have right biduals A° and B° in a Gray monoid $\mathcal{M}$. Let $f : A \to B$ and $g : B \to A$ be any arrows. Then the following two conditions are equivalent.*

(i) In $\mathbf{Sq}(\mathcal{M})$, $(g^\circ, e, n, \epsilon)$ is a right proto-bidual for f such that $\mathrm{Dom}(e) = e_A$, $\mathrm{Dom}(n) = n_A$, $\mathrm{Dom}(\epsilon) = \epsilon_A$, $\mathrm{Cod}(e) = e_B$, $\mathrm{Cod}(n) = n_B$ and $\mathrm{Cod}(\epsilon) = \epsilon_B$.

(ii) In $\mathcal{M}$, $(g, \tilde{e}, \vec{n})$ is a right proto-adjoint for f.

A right *coproto-adjoint* for $f : A \to B$ in $\mathcal{M}$ is a triple (g, λ, μ) such that (f, λ, μ) is a right proto-adjoint for g in $\mathcal{M}^{\mathrm{op}}$. Note that g is a right a adjoint for f if and only if there exist 2-cells λ and μ such that (g, λ, μ) is both a proto-adjoint and a co-proto adjoint for f.

A right *coproto-bidual* for X in $\mathcal{M}$ is a quadruple (Y, e, n, η) such that (X, n, e, η) is a right proto-bidual for Y in $\mathcal{M}^{\mathrm{coop}}$. Note that $(Y, e, n, \epsilon, \eta)$ is a right bidual for X if and only if (Y, e, n, ϵ) is a right proto-bidual for X and (Y, e, n, η) is a right coproto-bidual for X.

The following result is the 'co-' counterpart to Proposition 9, and is proved in an analogous way.

Proposition 10 *Suppose A and B have right biduals A° and B° in a Gray monoid $\mathcal{M}$. Let $f : A \to B$ and $g : B \to A$ be any arrows. The following two statements are equivalent.*

(i) *In* $\mathbf{Sq}(\mathcal{M})$, (g°, e, n, η) *is a right coproto-bidual for* f *such that* $\mathrm{Dom}(e) = e_A$, $\mathrm{Dom}(n) = n_A$, $\mathrm{Dom}(\eta) = \eta_A$, $\mathrm{Cod}(e) = e_B$, $\mathrm{Cod}(n) = n_B$ *and* $\mathrm{Cod}(\eta) = \eta_B$.

(ii) *In* $\mathcal{M}$, $(g, \tilde{e}, \vec{n})$ *is a right coproto-adjoint for* f.

Combining Propositions 3, 9 and 10, we obtain the desired theorem.

4 Compact closed Gray monoids and Ladj

We begin by recalling from [DS] the notion of a braided Gray monoid (though, for our purposes, we find it more natural consider structure 2-cell isomorphisms with the reverse orientation.) A *braiding* for a Gray monoid $\mathcal{M}$ consists of arrows $\rho_{X,Y} : X \otimes Y \to Y \otimes X$ which are equivalences for all objects X, Y and invertible 2-cells

$$\begin{array}{ccc} XY & \xrightarrow{\rho_{X,Y}} & YX \\ {\scriptstyle fg}\downarrow & \Downarrow\rho_{f,g} & \downarrow{\scriptstyle gf} \\ X'Y' & \xrightarrow[\rho_{X',Y'}]{} & Y'X' \end{array} \qquad \begin{array}{ccc} WXYZ & \xrightarrow{W X \rho_{Y,Z}} & WXZY \\ {\scriptstyle \rho_{W,X}YZ}\downarrow & \Downarrow\omega_{W,X,Y,Z} & \downarrow{\scriptstyle \rho_{W,XZ}Y} \\ XWYZ & \xrightarrow[X\rho_{WY,Z}]{} & XZWY \end{array}$$

satisfying a number of conditions which we wil not repeat here. We will call $\rho_{X,Y} : XY \to YX$ a braid arrow, $\rho_{f,g}$ a type-a-braid 2-cell and $\omega_{W,X,Y,Z}$ a type-b-braid 2-cell. A *braided Gray monoid* is a Gray monoid $\mathcal{M}$ together with a braiding for $\mathcal{M}$.

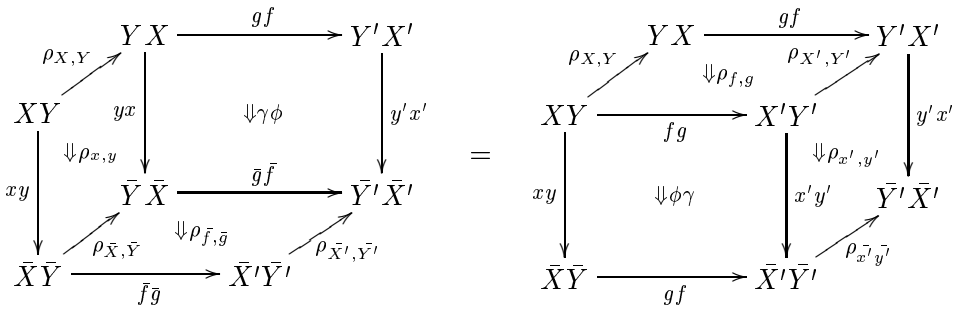
If $\mathcal{M}$ is a braided Gray monoid then we can also define a braiding for $\mathbf{Sq}(\mathcal{M})$. Suppose $x : X \to X'$ and $y : Y \to Y'$ are objects of $\mathbf{Sq}(\mathcal{M})$. Then the braid arrow $\rho_{x,y} : xy \to yx$ in $\mathbf{Sq}(\mathcal{M})$ is the type-a-braid 2-cell $\rho_{x,y}$ of $\mathcal{M}$. Suppose

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle x}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle x'} \\ \bar{X} & \xrightarrow[\bar{f}]{} & \bar{X}' \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ {\scriptstyle y}\downarrow & \Downarrow\gamma & \downarrow{\scriptstyle y'} \\ \bar{Y} & \xrightarrow[\bar{g}]{} & \bar{Y}' \end{array}$$

are squares in $\mathcal{M}$. The type-a-braid 2-cell

$$\begin{array}{ccc} xy & \xrightarrow{\rho_{x,y}} & yx \\ {\scriptstyle \phi\gamma}\downarrow & \Downarrow\rho_{\phi,\gamma} & \downarrow{\scriptstyle \gamma\phi} \\ x'y' & \xrightarrow[\rho_{x',y'}]{} & y'x' \end{array}$$

in $\mathbf{Sq}(\mathcal{M})$ is the unique 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathcal{M})$ satisfying $\mathrm{Dom}(\rho_{\phi,\gamma}) = \rho_{f,g}$ and $\mathrm{Cod}(\rho_{\phi,\gamma}) = \rho_{\bar{f},\bar{g}}$. So the type-a-braid 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathcal{M})$ naturally has the structure of a cube in $\mathcal{M}$:



Suppose $w : W \to \bar{W}$, $x : X \to \bar{X}$, $y : Y \to \bar{Y}$ and $z : Z \to \bar{Z}$, are objects of $\mathbf{Sq}(\mathcal{M})$. Then the type-b-braid 2-cell $\omega_{w,x,y,z}$ is defined to be the unique 2-cell of $\mathbf{Sq}(\mathcal{M})$ such that $\mathrm{Dom}(\omega_{w,x,y,z}) = \omega_{W,X,Y,Z}$ and $\mathrm{Cod}(\omega_{w,x,y,z}) = \omega_{\bar{W},\bar{X},\bar{Y},\bar{Z}}$.

That the structure 2-cells $\rho_{\phi,\gamma}$ and $\omega_{w,x,y,z}$ satisfy the conditions for a braiding follow from the fact that the 2-functors Dom and Cod are locally fully-faithful.

Recall from [DS] that a *right autonomous* Gray monoid is a monoidal bicategory such that every object has a right bidual. We define a *compact closed* Gray monoid to be a braided right autonomous Gray monoid. Note that if $\mathcal{M}$ is braided, the identity functor of $\mathcal{M}$ is equipped with a canonical structure of a monoidal homomorphism $\mathcal{M} \to \mathcal{M}^{\mathrm{rev}}$. It is clear then that a compact closed Gray monoid is also autonomous in the sense of [DS] (that is, every object has both a right and a left bidual).

Definition 1 If $\mathcal{M}$ is a Gray monoid then $\mathbf{Ladj}(\mathcal{M})$ is defined to be the full and locally-full subbicategory of $\mathbf{Sq}(\mathcal{M})$ comprising objects $f : A \to B$ such that f is a left adjoint (that is, has a right adjoint) as an arrow of $\mathcal{M}$. We call $\mathbf{Ladj}(\mathcal{M})$ the Gray monoid of left adjoints in $\mathcal{M}$.

As the tensor product of left adjoint arrows is still left adjoint and the opposite of a right adjoint arrow is a left adjoint, the following result follows from Theorem 1.

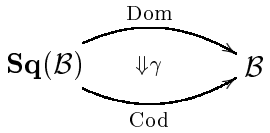
Corollary 1 *If $\mathcal{M}$ is a right autonomous (resp. compact closed) Gray monoid then* $\mathbf{Ladj}(\mathcal{M})$ *is right autonoumous (resp. compact closed).*

5 The bicategory of left adjoints

The higher dimensional algebraic structures that appear in nature tend to be 'weaker' than their lower dimensional counterparts. In particular, we find that monoidal bicategories – that is, one object tricategories – often arise in practice rather than Gray monoids. (Of course, a Gray monoid is a type of monoidal bicategory.) It is the case, however, that for any monoidal bicategory $\mathcal{B}$ there exists a Gray monoid $\mathcal{M}$ and strong monoidal biequivalence (that is, a triequivalence) $E : \mathcal{B} \to \mathcal{M}$. This follows from the coherence theorem for tricategories ([GPS]). In this section we extend the results of the previous section to the setting of monoidal bicategories.

We assume that the reader is familiar with the notion of an adjoint in a bicategory as well as a right bidual in a monoidal bicategory. Note that strong monoidal biequivalences preserve adjoints and right biduals. If $\mathcal{B}$ is a monoidal bicategory then we can define the monoidal bicategory $\mathbf{Sq}(\mathcal{B})$ of squares in $\mathcal{B}$ (which coincides with the earlier definition in the case of Gray monoids). Moreover, given any strong monoidal homomorphism $F : \mathcal{B} \to \mathcal{C}$ we can define a strong monoidal homomorphism $\mathbf{Sq}(F) : \mathbf{Sq}(\mathcal{B}) \to \mathbf{Sq}(\mathcal{C})$. If F is a biequivalence then so is $\mathbf{Sq}(F)$. Also, the notions of a right autonomous, braided and compact closed Gray monoid can be extended in the obvious way to the setting of monoidal bicategories.

For any $\mathcal{B}$, there exists an op-lax transformation



where $\gamma_f = f$. For any homomorphism $F : \mathcal{B} \to \mathcal{C}$, we have that $\mathbf{Sq}(F) \cdot \mathrm{Dom}_{\mathcal{C}} = \mathrm{Dom}_{\mathcal{B}} \cdot F$, $\mathbf{Sq}(F) \cdot \mathrm{Cod}_{\mathcal{C}} = \mathrm{Cod}_{\mathcal{B}} \cdot F$ and $\mathbf{Sq}(F) \cdot \gamma_{\mathcal{C}} = \gamma_{\mathcal{B}} \cdot F$. The extension of Theorem 1 is now straightforward.

Theorem 2 *Let $f : A \to B$ be an arrow of a monoidal bicategory $\mathcal{B}$. The object f of $\mathbf{Sq}(\mathcal{B})$ has a right bidual if and only if (i) both A and B have right biduals in $\mathcal{B}$ and (ii) the arrow f has a right adjoint in $\mathcal{B}$.*

Proof Let $\mathcal{B}$ be a monoidal bicategory and let $E : \mathcal{B} \to \mathcal{M}$ be a strong monoidal biequivalence with codomain a Gray monoid. Let $f : A \to B$ be an arrow of $\mathcal{B}$ – that is, an object of $\mathbf{Sq}(\mathcal{B})$. Since E and $\mathbf{Sq}(E)$ are biequivalences, we have that the following four conditions are equivalent.

The object f has a right bidual.

The object $\mathbf{Sq}(E)(f)$ has a right bidual.

(i) $\mathrm{Dom}(\mathbf{Sq}(E)(f))$ and $\mathrm{Cod}(\mathbf{Sq}(E)(f))$ have right biduals and (ii) $\gamma_{\mathbf{Sq}(E)(f)}$ has a right adjoint.

(i) A and B have right biduals and (ii) f has a right adjoint.

□

Corollary 2 *If $\mathcal{B}$ is a right autonomous (resp. compact closed) bicategory then* $\mathbf{Ladj}(\mathcal{B})$ *is right autonomous (resp. compact closed).*

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